# Sin(x)/x Interpolation: An Important Aspect of Proper Oscilloscope Measurements

By Chris Rehorn, Agilent Technologies

Digitizing real-time oscilloscopes provide the backbone of high speed time-domain measurements made in the industry today. Modern oscilloscopes use high-speed digitizers to capture the input signal. An oscilloscope's sample rate is often touted as a banner specification for the instrument; higher being better. In reality, as long as the rules of Nyquist are not violated, an oscilloscope can reconstruct a user's signal *identically*. This reconstruction process is often referred to as *sin(x)/x* interpolation. Whether the sample rate is 25x the Nyquist frequency, or 2.5x the Nyquist frequency, interpolation can be used to reproduce the waveform exactly as it appeared at the oscilloscopes input connector, removing all doubt about a signal's behaviour between samples.

## Nyquist Revisited

One of the fundamental concepts taught in signal processing courses is the concept of sampling. In its simplest form, sampling is the process by which a continuous time signal is "sampled" at discrete points in time, rendering a collection of samples that create a discrete time signal. Dr. Harry Nyquist's most famous theorem proposes that a signal can be reconstructed perfectly from discrete samples if the following two rules are observed:

1. The highest frequency component sampled must be less than half the sampling frequency and
2. Samples must be acquired in equally spaced intervals.

These concepts are familiar to most engineers and are by no means new or novel. Their application, however, is of utmost importance in understanding how digitizing oscilloscopes reconstruct the user's signal from discrete samples.

## Interpolation Basics

Sampling is a relatively straight-forward process in which a continuous time signal is converted into a discrete time signal. The method by which the continuous time signal is reconstructed from its samples is just as important. Although there are many methods to perform signal reconstruction, *sin(x)/x* interpolation is the method used by most modern-day oscilloscopes. This interpolation method is of significant value because the original waveform can be identically reconstructed using this method.

Assume $x(t)$ is a time-domain waveform. If $x(t)$ is sampled by a pulse train, $s(t)$ then the sampled signal $x_x(t)$ can be expressed as:

$$x_s(t) = x(t) * s(t) \tag{1}$$

where the sample pulse train can be expressed as

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{2}$$

The expression in (2) is a simple pulse train of infinite length. This pulse train is shown in **Figure 1**, for positive values of $n$.

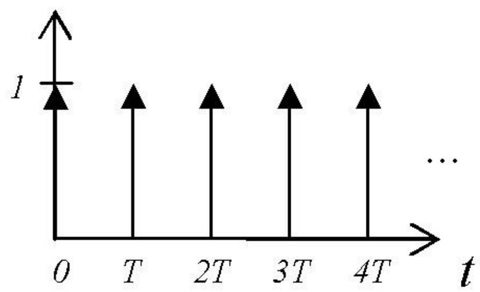


**Figure 1: Sample pulses shown in the time domain. Note *T* is the sample period.**

Recalling basic signal processing theory, multiplication in the time domain is equivalent to convolution in the frequency domain.

$$X_s(f) = X(f) \otimes \ '(f) \tag{3}$$

The Fourier Transform of the impulse train can be represented by an infinite sum of impulses in the frequency domain, scaled by the inverse of the sample period. This is expressed in (4) and shown in **Figure 2**.

$$S(f) = \frac{1}{T} \sum_{m= \ \infty}^{\infty} \delta \ f - \frac{m}{T}). \tag{4}$$

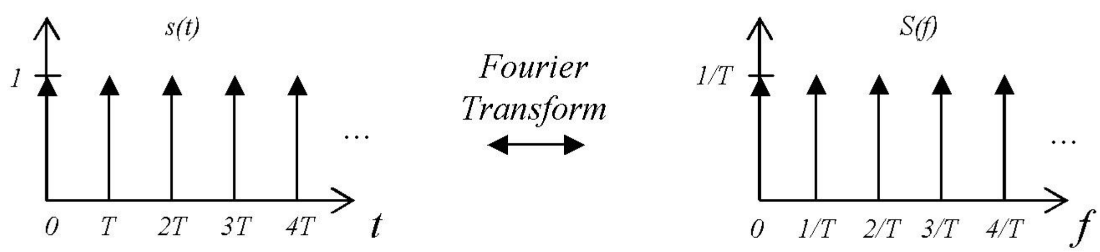


**Figure 2: The Fourier Transform of a pulse train in the time domain is a pulse train the frequency domain, separated by integer multiples of *1/T*, where T is the sample period used to sample the original analog waveform.**

The value of *1/T* is deemed the sample frequency, or $f_s$. The convolution given in (3) is depicted graphically in **Figure 3**. Only the magnitude spectra has been shown for the various signals.

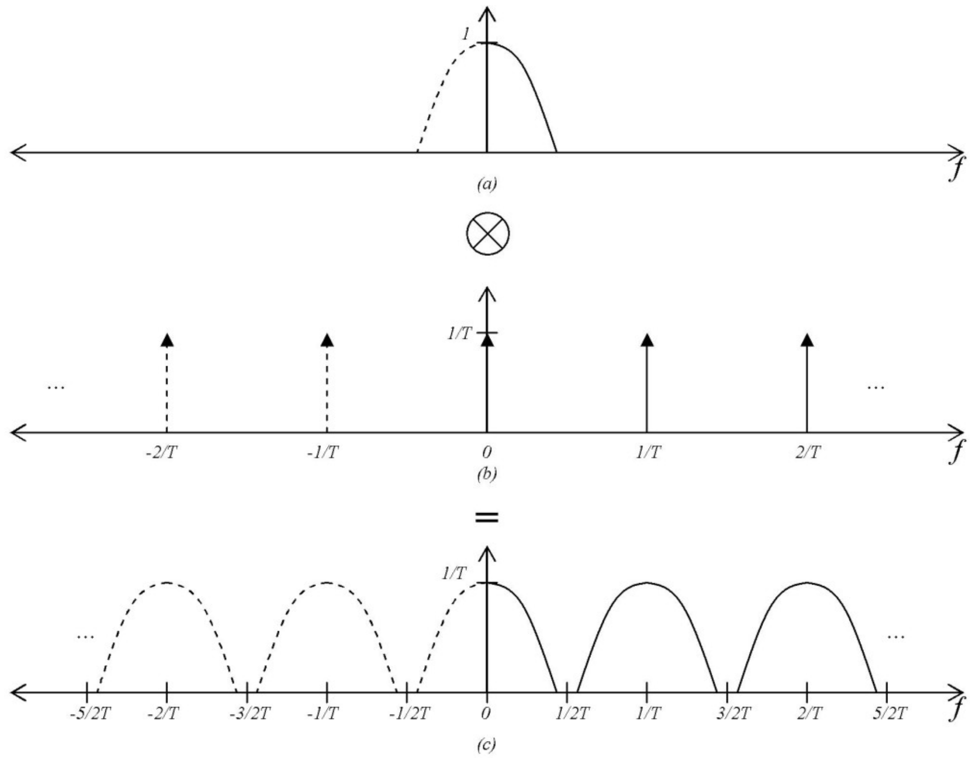


**Figure 3: Convolution of the input signal *X(f)* (*a*) with the sample pulse train *S(f)* (*b*) produces a frequency spectrum magnitude shown in (*c*). Note that (*c*) is simply a repetition of the spectrum shown in (*a*) at integer multiples of *1/T*. The value of *1/T* is referred to as the sampling frequency, or $f_s$.**

Figure 3 clearly indicates that the sampled waveform frequency spectrum is simply a repetition of the original waveform spectrum by integer multiples of $f_s$. This figure indicates that for larger values of $T$, the sampling period, the repeating frequency spectra will begin to overlap each other. This is what is referred to as *aliasing*. This effect is shown in **Figure 4**.

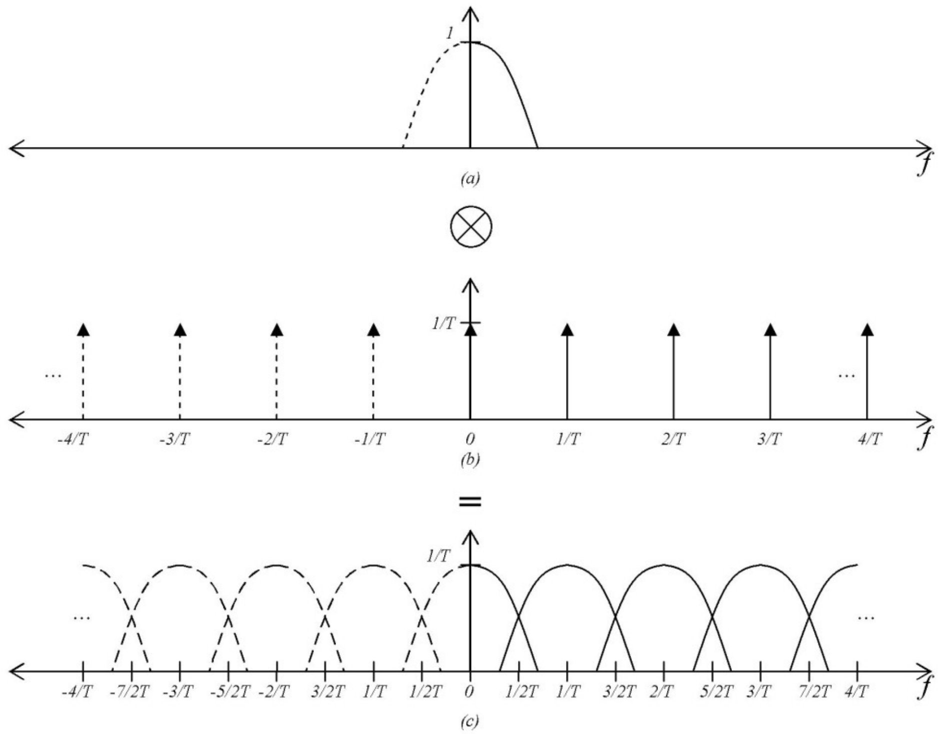


**Figure 4: Aliasing occurs when *T*, the sample period, becomes excessively large. Convolving (a) with (b) produces the frequency spectrum shown in (c). High frequency components of the image spectrum "impose" themselves on low frequency components of the original signal spectrum.**

If the spectrum of the analog signal input has frequency content above $f = 1/(2T)$ the signal will alias, as shown in Figure 4. This frequency has a special name. It is known as the *Nyquist Frequency* can be expressed as

$$f_N = \frac{f_s}{2}. \tag{5}$$

Assuming that the analog input signal has no frequency content above $f_N$, the original signal content can be reconstructed from the discrete time waveform. By rejecting the higher frequency content above $f_N$ in Figure 3 (c), the original analog waveform spectrum is recovered. Compare this scenario with that shown in Figure 4. The original signal spectrum cannot be recovered by simply rejecting frequencies above $f_N$ because the spectrum has been corrupted after sampling. An ideal rectangular filter will accomplish this task for the non-aliased case. The filter simply accepts all frequency content below $f_N$ and perfectly rejects all higher frequency content. By multiplying the spectrum of the sampled signal with the rectangular filter, the original spectrum is recovered. This is depicted in **Figure 5**.

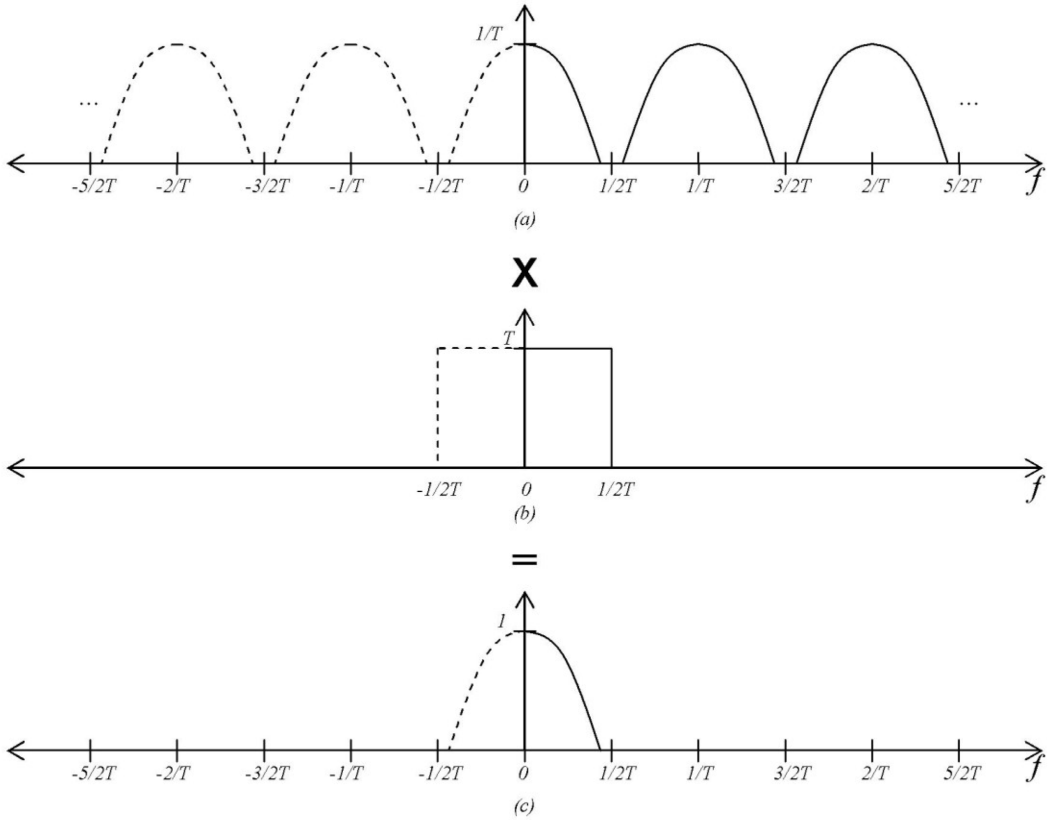


**Figure 5: Multiplying the sampled spectrum (*a*) by an ideal rectangular filter (*b*) yields the original analog signal spectrum (*c*). Notice that the low-pass filter has infinite slope at the Nyquist frequency. This is a lot of poles, zeroes, or taps, for the DSP-savvy readers.**

Perhaps the most amazing concept behind this type of reconstruction is that the original waveform is *exactly* recovered from the sampled waveform using the rectangular interpolation filter. This interpolation method is what is often referred to as *sin(x)/x*, or *sinc(x)* interpolation. So where does this name come from?

Recall that multiplication of signals in the frequency domain is equivalent to convolving signals in the time domain. Also, recall that the Inverse Fourier Transform of a rect function in the frequency domain, is a sinc function in the time domain. The rect function shown in Figure 5 (*b*) can be expressed as a sinc function in the time domain:

$$r(t) = \frac{\sin(f_s \pi)}{f_s \pi}. \tag{6}$$

And, finally, the interpolated waveform can be expressed in terms of the sampled waveform and the *sin(x)/x* interpolation filter:

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] * \mathrm{sinc}(f_s(t - nT)) \tag{7}$$

where *x[n]* is the set of discrete time samples obtained during the sampling process, and has a spectrum identified by that shown in Figure 5 (*a*).

## Practical Implementation of sin(x)/x Interpolation

The mathematics behind the *sin(x)/x* interpolation, while elegant, lend little help in actual implementation of the algorithm. The interpolation method requires an infinite amount of sample points before and after the $n^{th}$ point being interpolated. In reality, however, FIR filters can be designed to closely replicate the *sin(x)/x* reconstruction filter, with a finite number of samples. This is precisely what Agilent oscilloscopes implement while acquiring a user's signal. Once the signal is digitized, the discrete samples are passed through various filters; one of these being an interpolation filter used to reconstruct the waveform on screen.

In modern oscilloscopes, it is not unheard of to see instruments boasting sampling rates 25x higher than the specified oscilloscope bandwidth. Often this gives customers a feeling they are not "missing" events occurring between samples. From the mathematics shown above, however, as long as the Nyquist criteria are not violated, a sampling rate of $2\,f_N$ is sufficient!

Engineers often worry about glitches occurring between samples that are not acquired by the oscilloscope. If a glitch occurs between samples, however, the input signal exceeds the bandwidth of the instrument being used. With either an analog *or digital* oscilloscope, the signal path will low-pass filter this glitch, spreading its energy out in time and reducing its amplitude. This bandwidth limited glitch will be sampled by the digital oscilloscope and will be seen no better or worse than on an analog oscilloscope.

So, why aren't oscilloscope sampling rates exactly 2x higher than the oscilloscope's bandwidth? This is because out-of-band signals may only be attenuated by 10 dB beyond the band-edge of the instrument. Said differently, the frequency response of the oscilloscope does not roll-off infinitely fast and some buffer room is used on the sampling rate to minimize aliasing. More on signal aliasing later.

A practical example *of sin(x)/x* reconstruction is shown below. A simple script was written to create a) an analog signal, b) a sampled version of the analog signal and c) a new waveform of data points reconstructed from the sample points using *sin(x)/x* interpolation. In each figure, the analog waveform is shown in blue, the sampled waveform shown in red, and the interpolated data points are shown with black triangles.

Suppose a 1 GHz sinusoidal wave is digitized at 25 GS/s. This is a factor 12.5 higher in sampling rate than what is ideally needed for proper reconstruction. This scenario is shown in Figure 6. Note that the red stars correspond to samples taken of the sinusoid. The blue waveform represents the original analog signal, and the black triangles represent the waveform reconstructed from *sin(x)/x* interpolation.

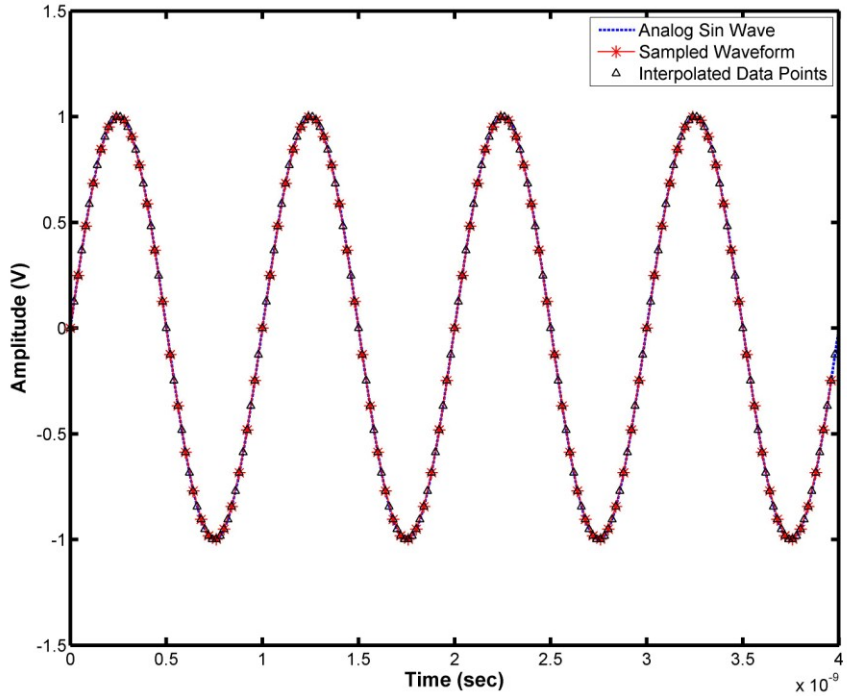


**Figure 6: When a 1 GHz sinusoidal waveform is digitized at 25x the fundamental frequency, "connect-the-dots" interpolation is hard to separate from the ideal reconstruction, and the original waveform.**

The waveforms shown in **Figure 6** are visually appealing to an oscilloscope user because it is easy to associate the samples with the original signal. Even if the interpolation algorithm is disabled by the oscilloscope user, the samples themselves accurately portray the signal that was present at the instrument's input.

Suppose the sampling rate is reduced by an order of magnitude to 2.5x the original signal's frequency, or 2.5 GS/s. This scenario is shown in **Figure 7**. Even at this lower sample rate, the Nyquist Theorem is not violated. The red stars mark the sample locations of the original signal. "Connect-the-dots" reconstruction is shown along with *sin(x)/x* interpolated data.

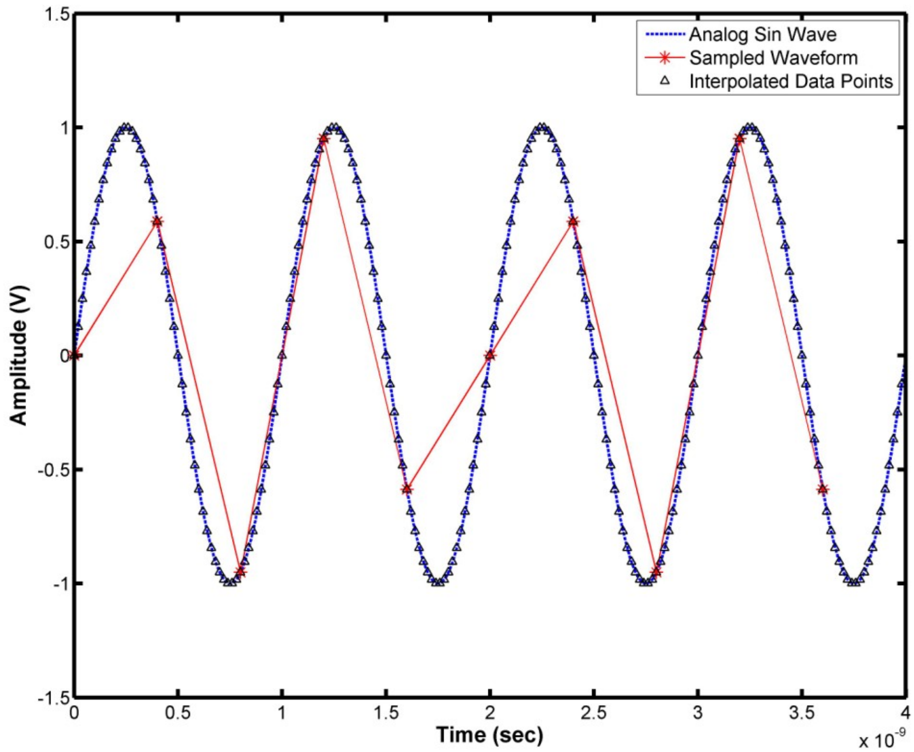


**Figure 7: When the sampling rate is only 2.5x the highest frequency, it is less obvious to the eye what the analog waveform looked like before sampling. Provided the Nyquist criteria are maintained, *sin(x)/x* interpolation can be used to identically reproduce the waveform, just as it was when the sample rate was 10 times higher.**

The reduced sample rate makes it less obvious what waveform existed from the samples extracted. *Sin(x)/x* interpolation reconstructs the analog waveform true to its original form, even at this drastically reduced sample rate.

Most oscilloscope users are not measuring ideal sinusoidal tones. **Figure 8** shows an ideal square wave consisting of the first, third and fifth harmonics of a 200 MHz square wave. The waveform is sampled at 2.5 GS/s, exactly 2.5 times faster than the fifth harmonic of the waveform.

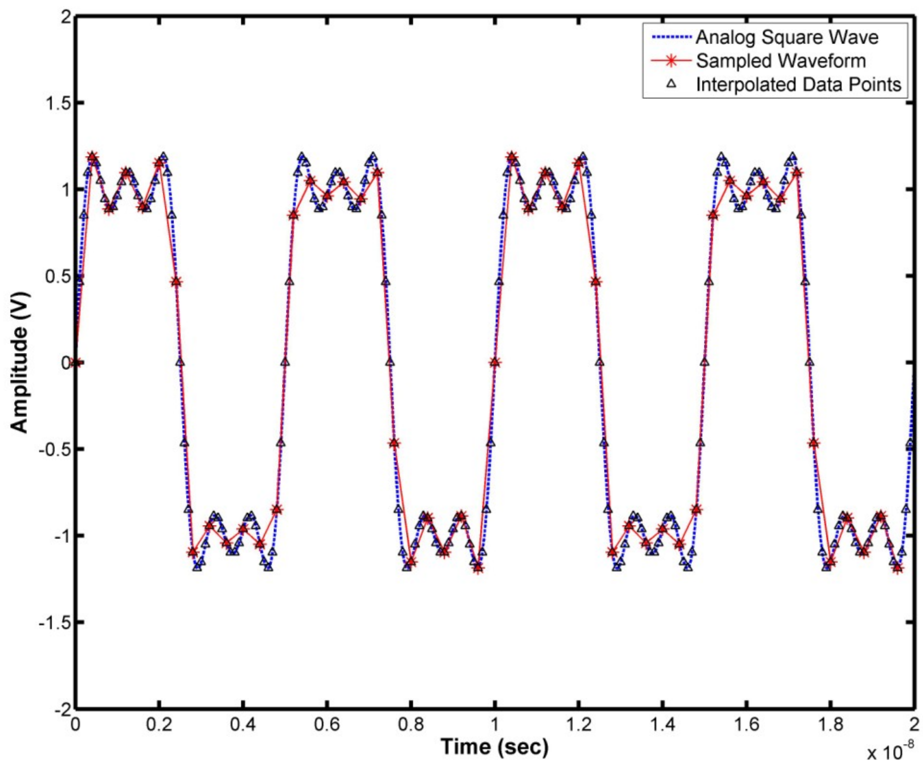


**Figure 8: A 200 MHz square wave consisting of the 1st, 3rd, and 5th harmonics can be ideally reconstructed using *sin(x)/x* interpolation, as long as the Nyquist criteria are maintained during sampling.**

Figure 8 clearly shows that the *sin(x)/x* reconstruction filter perfectly reconstructs the waveform from the samples taken at a rate 2.5 times faster than the highest frequency content of the waveform.

## A Note about Aliasing

The above cases show interpolation on signals that do not violate the Nyquist criteria. If, however, a signal with significant frequency content above $f_N$ is digitized, aliasing will occur. In this case, it is no longer possible to reconstruct the original waveform exactly as it existed before sampling because high frequency components of the original signal now appear as lower frequency components after the sampling process has occurred. This is certainly not acceptable in a high-fidelity system.

To prevent aliasing, two fundamental items can be addressed in a digitizing system:

1) An anti-alias filter can be designed to reject all high-frequency content above $f_N$.
2) The sample rate can be increased to an arbitrarily high frequency such that $f_N$ is above the highest significant frequency component of the signal being digitized.

In some systems, one might imagine a case where increased sample rate is inexpensive, compared to a higher-order anti-alias filter. In this case, it may be more practical to increase the sample rate to compensate for the slow roll-off of the anti-alias filter, instead of designing a more complex filter.

For more information about aliasing, and its effect on oscilloscope measurements, please refer to Agilent Application Note 1587, available from http://www.agilent.com.

## Make Some Measurements…

An oscilloscope with more sampling rate is not always better. The signal at the input of the oscilloscope is properly reconstructed after digitization by the *sin(x)/x* reconstruction filter, if the input signal does not contain frequency content above and beyond the Nyquist frequency. It may be tempting to disable interpolation to see the "raw samples", but this is not necessary. The interpolated waveform is not a "guess" at what the signal was doing between samples, it *is exactly* what the signal was doing between samples. So now, go out in confidence with interpolation enabled on your instrument, and make some measurements, because that is the fun part anyway!

**About the author**

**Chris Rehorn,** *Analog ASIC Design Engineer,* Digital Test Division, Electronic Measurements Group, Agilent Technologies

***Chris Rehorn*** is an analog circuit design engineer with Agilent Technologies. His work focuses on the design and implementation of analog circuits for Agilent's high-performance oscilloscopes division. Prior to his position at Agilent, Chris worked on microwave and RF hardware design, with specific focus on superheterodyne receivers.

Chris Rehorn joined Agilent Technologies in 2006 after obtaining both a bachelor's degree in electrical engineering and his master's of science degree in electrical engineering from the University of Virginia, Charlottesville Va. In his spare time he enjoys playing jazz guitar, playing the tabla, running, hiking and mountain biking in Colorado, with his wife.