An Analysis of Certain Errors in Electronic Differential Analyzers
II—Capacitor Dielectric Absorption*

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Summary—**The permittivity or dielectric constant of the materials used in capacitors is not actually a constant but is a complex function of frequency and temperature. Consequently, the feedback capacitors used in the integrators of a differential analyzer cannot be considered ideal, but their capacitance must be considered a variable. Methods of representing the complex capacitance are discussed and a model is selected which is conveniently suited to the analysis. Experimental methods of measuring the complex capacitance are described. The phenomenon of dielectric absorption is interpreted in terms of the capacitor model and it is shown that an integrator having such a feedback capacitor will experience a change in effective initial conditions after a solution is started on the computer. It is also shown that when such integrators are used to solve linear differential equations with constant coefficients, the locations of the roots of the characteristic equation are changed slightly; these changes can be evaluated when the properties of the capacitor model are known.**

INTRODUCTION

A well-known phenomenon exhibited by dielectric materials is that of dielectric absorption.[1] When a potential difference is applied to a dielectric, the polarization current or charging current consists of two distinct types. The first is the charging current which occurs practically instantaneously; the second is that which occurs more slowly during a measurable period of time. The former is caused by the rapidly forming, or instantaneous, electronic and atomic polarizations. The latter is caused by slowly forming, or absorptive, dipole and interfacial polarizations. Only the interfacial polarizations have relaxation times which are large enough to be of interest in analog computer applications. (The relaxation time is a quantitative measure of the time required for a polarization to form or disappear.)

The theory of dipole polarizations as developed by Debye[2] shows that the dielectric constant can be considered a complex function defined by

$$\epsilon^* = \epsilon' - j\epsilon''. \tag{1}$$

When expressed in terms of frequency and relaxation time, the complex dielectric constant becomes

$$\epsilon^* = \epsilon_\infty + \frac{\epsilon_0 - \epsilon_\infty}{1 + j\omega\tau_0} \tag{2}$$

where ϵ_0 is the zero frequency or static dielectric constant, ϵ_∞ is the infinite frequency dielectric constant, and τ_0 is the relaxation time which is a function of temperature.

The theory of interfacial polarizations has been treated by several writers,[3–7] and various expressions for the complex dielectric constant have resulted, all determined in part by empirical parameters. Of these, perhaps the simplest is that proposed by Cole and Cole,[7] which is

$$\epsilon^* = \epsilon_\infty + \frac{\epsilon_0 - \epsilon_\infty}{1 + (j\omega\tau_0)^{1-\alpha}} \tag{3}$$

where α is an empirical constant with values from 0 to 1, a measure of the distribution of relaxation times.

Unfortunately, none of these expressions, nor the physical model of the capacitor resulting from them, is particularly well suited to the analysis of the effect of such a capacitor when used as the feedback path of a high-gain amplifier, *i.e.*, when used in an integrator of a differential analyzer. Furthermore, these expressions were based on the results of experiments performed on dielectric materials not commonly used in the capacitors used in analog computers. Consequently, the experiments described below were carried out for two purposes: 1) to develop a useful mathematical expression for the complex capacitance and a physical model for the capacitor, and 2) to evaluate the necessary empirical parameters for a dielectric material commonly employed in precision computer capacitors, namely, polystyrene.

TRANSIENT CURRENT MEASUREMENTS

For the measurement of the charging current, an electrometer was connected in series with the capacitor and a battery. The charge on the electrometer was re-

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[1] E. J. Murphy and S. O. Morgan, "The dielectric properties of insulating materials," *Bell Sys. Tech. J.*, vol. 16, pp. 493–512; October, 1937.

[2] P. Debye, "Polar Molecules," Chemical Catalog Company, New York, p. 94; 1929.

[3] J. C. Maxwell, "Electricity and Magnetism," Oxford University Press, London, Eng., 3rd ed., vol. 1, ch. 10; 1892.

[4] K. W. Wagner, "Zur theorie der unvollkommenen dielektrika," *Ann. der Physik*, vol. 40, pp. 817–855; 1913.

[5] W. A. Yager, "The distribution of relaxation times in typical dielectrics," *Physics*, vol. 7, pp. 434–450; December, 1936.

[6] R. M. Fuoss and J. G. Kirkwood," Electrical properties of solids, VIII," *J. Amer. Chem. Soc.*, vol. 63, pp. 385–394; February, 1941.

[7] K. S. Cole and R. H. Cole, "Dispersion and absorption in dielectrics—I. Alternating current characteristics," *J. Chem. Phys.*, vol. 9, pp. 341–351; April, 1941.

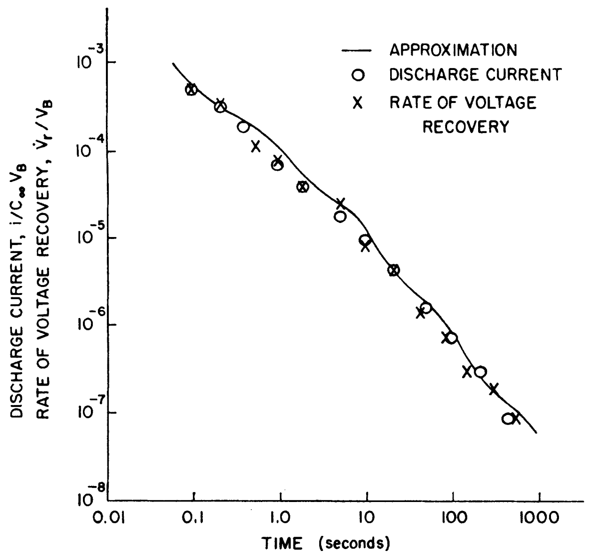

Fig. 1—Capacitor discharge current and rate of voltage recovery.

corded, and its rate of change evaluated to give the current. The discharge current was measured in a similar manner by replacing the battery by a short circuit. The charging current was influenced by the capacitor leakage resistance and by the battery voltage recovery following the initial surge of current. Since the discharge current was not subject to these influences, the charging current will not be considered in what follows.

It was noted that, due to the dielectric absorption effect, the magnitude of the discharge current depended on how long the capacitor was charged prior to being discharged. If the capacitor was charged for more than fifteen minutes, the discharge current during the period of observation (about five minutes) stabilized and did not change when a longer charging interval was used.

The data points in Fig. 1 show the measured discharge current for a 1-μf polystyrene capacitor following a fifteen-minute charging interval. Initial charging voltages of from 23 to 110 volts were used with results essentially identical to those shown in Fig. 1. No difference in results was noted for temperatures ranging from 66°F to 102°F.

Any number of functions could be selected to fit the data in Fig. 1. The function selected was

$$\frac{i}{C_\infty V_B} = \sum_k a_k e^{-t/T_k} \tag{4}$$

where the number of terms would be determined by the accuracy required in the approximation.

In Fig. 2 the current i, following the application of voltage V_B with the capacitors initially discharged, is given by

$$\frac{i}{C_\infty V_B} = \sum_k \frac{1}{R_k C_\infty} e^{-t/R_k C_k}. \tag{5}$$

The discharge current which would flow if the terminals in Fig. 2 were shorted after all the capacitors are

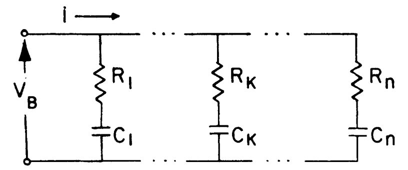

Fig. 2—Model of absorptive portion of capacitor.

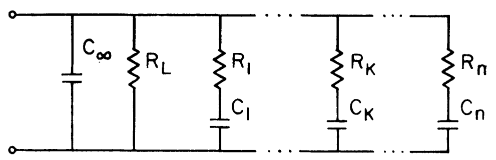

Fig. 3—Model of capacitor based on experimental results.

charged to V_B volts is, of course, also given by (5). Eqs. (4) and (5) will be the same if the resistors and capacitors have values given by the following relations:

$$R_k C_\infty = \frac{1}{a_k}$$
$$\frac{C_k}{C_\infty} = T_k a_k. \tag{6}$$

It follows directly then that the equivalent circuit for the capacitor is as shown in Fig. 3, where the leakage resistance R_L is included. A similar model has also been suggested by Single.[8]

It is possible, using five resistor-capacitor combinations, to fit the data given in Fig. 1 if the values used for the components in Fig. 3 are those given in Table I. The discharge current which would result from this approximation is plotted in Fig. 1 for comparison with the experimentally observed discharge current.

TABLE I
EXPERIMENTAL VALUES FOR CAPACITOR MODEL

k	C_k/C_∞	R_kC_∞ (seconds)	$T_k=R_kC_k$ (seconds)
1	1.40×10^{-4}	3.56×10^{6}	500
2	2.00×10^{-4}	2.50×10^{5}	50
3	2.70×10^{-4}	2.00×10^{4}	5.4
4	1.93×10^{-4}	3.03×10^{3}	0.585
5	1.20×10^{-4}	3.34×10^{2}	0.040

$R_LC_\infty = 5.0\times10^6$

Complex Capacitance

The complex capacitance C^* of the model in Fig. 3 can be evaluated from its admittance since, for a capacitor,

$$Y(j\omega) = j\omega C^*. \tag{7}$$

From Fig. 3, the admittance is seen to be

$$Y(j\omega) = j\omega C_\infty + \sum_k \frac{1}{R_k + \dfrac{1}{j\omega C_k}} \tag{8}$$

[8] C. H. Single, "Precision Components for Analog Computers," paper presented at ISA Convention, New York, N. Y.; 1956.

where the admittance due to the leakage resistance is not included since it is not part of the polarization process.

The complex capacitance from (7) and (8) is

$$C^*(j\omega) = C_\infty + \sum_k \frac{C_k}{1 + j\omega T_k} \tag{9}$$

where T_k is the kth relaxation time, R_kC_k. From (9) it is clear that the infinite frequency capacitance is

$$C_\infty = C^*]_{\omega=\infty} \tag{10}$$

and the zero frequency capacitance is

$$C_0 = C^*]_{\omega=0} = C_\infty + \sum_k C_k. \tag{11}$$

The complex capacitance can also be expressed in terms of its real and imaginary parts as

$$C^* = C' - jC'' \tag{12}$$

where

$$C' = C_\infty + \sum_k \frac{C_k}{1 + \omega^2 T_k^2} \tag{13}$$

$$C'' = \sum_k \frac{\omega C_k T_k}{1 + \omega^2 T_k^2}\cdot \tag{14}$$

If C^* is expressed in polar coordinates, one has

$$C^* = |C^*| e^{-j\delta} \tag{15}$$

where

$$|C^*| = \sqrt{C'^2 + C''^2} \tag{16}$$

$$\tan \delta = \frac{C''}{C'}\cdot \tag{17}$$

Tan δ is the loss tangent or dissipation factor of the dielectric. If $\sum_k C_k \ll C_\infty$, the following approximate expressions can be derived:

$$|C^*| = C_\infty + \sum_k \frac{C_k}{1 + \omega^2 T_k^2} \tag{18}$$

$$\delta = \frac{1}{C_\infty} \sum_k \frac{\omega C_k T_k}{1 + \omega^2 T_k^2}\cdot \tag{19}$$

Eq. (19) is plotted in Fig. 4 in dashed lines for the capacitor model given by Table I.

Cole and Cole, in a second paper,[9] have evaluated the transient current following the application of a constant voltage to a capacitor having a dielectric constant given by (3). These results can be made to agree with the experimental discharge current of Fig. 1 if the parameters have the following values:

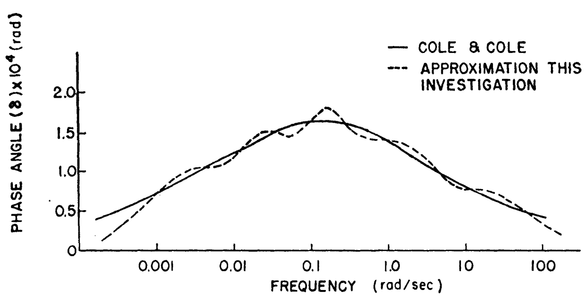

Fig. 4—Phase angle of the complex capacitance vs frequency.

$$\alpha = 0.6$$

$$\tau_0 = 7.0 \text{ seconds}$$

$$\frac{C_0 - C_\infty}{C_\infty} = 10^{-3}. \tag{20}$$

The details of the method used for matching the transient current curves are given by Field.[10]

The value of the dissipation factor, δ, as given by Cole and Cole[9] using the parameters given in (20) is plotted in Fig. 4 in a solid line for comparison with (19). Although the method of Cole and Cole permits the complex capacitance to be described by fewer parameters than does the method used here, the latter method lends itself more readily to the error analysis which is the purpose of this report.

Capacitor Voltage Recovery

If a capacitor is charged at a constant voltage for a period of time and then briefly discharged by short circuiting its terminals, a voltage is observed to build up on the open-circuited terminals due to the slow forming polarizations. This behavior provides a convenient means of measuring the same properties of the dielectric which were determined by the transient current measurements described above.

If the model of the capacitor in Fig. 3 is charged with a voltage V_B for a length of time, which is long compared to the longest relaxation time R_kC_k, then each of the capacitors C_k will be charged to essentially V_B volts. The voltage V_B is removed and the capacitor terminals are short-circuited for a length of time, which is short compared to the shortest relaxation time, thus reducing the voltage on C_∞ to zero and leaving the voltage on the other capacitors still essentially at V_B. The short circuit is removed and the capacitor terminals are left open-circuited. A time varying voltage will now appear on the terminals. Let this recovery voltage be defined as V_r.

If the leakage resistance R_L is large enough for a negligible portion of the charge on C_∞ to leak off during the voltage recovery (*i.e.*, if $R_LC_\infty \gg R_kC_k$) and if, in

[9] K. S. Cole and R. H. Cole, "Dispersion and absorption in dielectrics—II. Direct current characteristics," *J. Chem. Phys.*, vol. 10, pp. 98–105; February, 1942.

[10] R. F. Field, "Dielectric Measuring Techniques, Permittivity, Lumped Circuits" in "Dielectric Materials and Applications," A. R. von Hippel, ed., John Wiley and Sons, New York, N. Y., pp. 47–62; 1954.

addition, $C_k/C_\infty \ll 1$, analysis of Fig. 3 gives for the recovery voltage

$$V_r \cong \frac{V_B}{C_\infty} \sum_k C_k(1 - e^{-t/R_kC_k}). \tag{21}$$

Differentiating (21) with respect to time and rearranging yields

$$\frac{\dot{V}_r}{V_B} = \sum_k \frac{1}{R_kC_\infty} e^{-t/R_kC_k}. \tag{22}$$

Comparison of (5) and (22) shows that

$$\frac{\dot{V}_r}{V_B} = \frac{i}{C_\infty V_B} \tag{23}$$

where V_r is the recovery voltage and i is the discharge current. Thus the measurement of the capacitor recovery voltage can be used to obtain the same information as that obtained from current discharge measurements.

The recovery voltage can be measured by connecting the capacitor to a very high impedance voltmeter, such as an electrometer. An alternative method is to use the capacitor as the feedback of a high-gain computer amplifier. With the computer in the operating condition, the capacitor is discharged by short-circuiting its terminals. The build-up of voltage on the capacitor will then appear as the output voltage of the high-gain amplifier. The advantage of this method is that the capacitor characteristics can be determined without removing it from the computer and without additional equipment. A disadvantage is the integrator drift due to grid current. This effect can be minimized, however, by making two tests using the same amplifier, the second test with the charging voltage equal in magnitude but opposite in sign to the first. When the results of the two tests are subtracted, the integrator drift will be eliminated.

Results of a typical experimental measurement of the voltage recovery, using the same polystyrene capacitor as before, are shown in Fig. 1.

Integrator Errors

Fig. 5 shows a high-gain operational amplifier connected as an integrator with a feedback capacitor having dielectric absorption represented by the model developed earlier. Assuming that the amplifier is ideal[11] (*i.e.*, it has infinite gain), the following equations can be written:

$$i_i = i_c + \sum_k i_k \tag{24}$$

$$i_i = \frac{V_i}{R_i} \tag{25}$$

[11] For a discussion of the effect of nonideal amplifiers see P. C. Dow, Jr., "An analysis of certain errors in electronic differential analyzers—I. Bandwidth limitations," IRE Trans. on Electronic Computers, vol. EC-6, pp. 255–260; December, 1957.

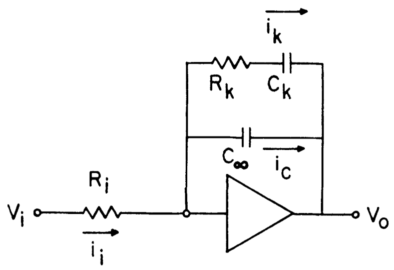

Fig. 5—Amplifier connected as an integrator.

$$i_c = -C_\infty \frac{dV_0}{dt} \tag{26}$$

$$R_k \frac{di_k}{dt} + \frac{i_k}{C_k} = -\frac{dV_0}{dt}. \tag{27}$$

Taking the Laplace transform of (24)–(27) and eliminating the current terms yields

$$\left[C_\infty + \sum_k \frac{C_k}{R_kC_ks + 1}\right][sV_0(s) - V_0(0)] = -\frac{V_i(s)}{R_i} + \sum_k \frac{R_kC_ki_k(0)}{R_kC_ks + 1} \tag{28}$$

where

$$V_0(s) = \mathcal{L}[V_0(t)]$$
$$V_0(0) = V_0(t)]_{t=0}. \tag{29}$$

Making use of (9) in (28) gives

$$V_0(s) = -\frac{V_i(s)}{sR_iC^*(s)} + \frac{V_0(0)}{s} + \frac{1}{sC^*(s)} \sum_k \frac{R_kC_ki_k(0)}{R_kC_ks + 1}. \tag{30}$$

The first term in (30) will give the output of the integrator resulting from a given input; the second and third terms give the output resulting from the initial conditions.

Consider the response of the integrator to a unit impulse. If $R_iC_\infty = 1$, the first term of (30) becomes

$$V_0(s) = -\frac{1}{s\left[1 + \sum_k \frac{C_k/C_\infty}{1 + T_ks}\right]}. \tag{31}$$

If $\sum_k C_k/C_\infty \ll 1$, (31) can be written approximately as

$$V_0(s) = -\frac{1}{s}\left[1 - \sum_k \frac{C_k/C_\infty}{T_ks + 1}\right]. \tag{32}$$

Taking the inverse Laplace transform of (32) yields

$$V_0(t) = -1 + \sum_k \frac{C_k}{C_\infty}(1 - e^{-t/T_k}). \tag{33}$$

The last term of (33) is the voltage recovery to a unit applied voltage, as given by (21). Then (33) can be written

$$V_0(t) = -1 + \frac{V_r(t)}{V_B} \tag{34}$$

where $V_r(t)$ is the recovery voltage following an applied voltage V_B.

That is, the effect of dielectric absorption is to reduce the magnitude of the desired integrator impulse response by the amount of the capacitor voltage recovery.

Similarly, it can be shown that if

$$V_i(t) = R_i C_\infty \cos \omega t, \tag{35}$$

the integrator output is approximately

$$V_0(t) = -\frac{C_\infty}{\omega |C^*|} \sin(\omega t + \delta) \tag{36}$$

where $|C^*|$ and δ are defined by (18) and (19).

The second term in (30) is the normal initial condition of the integrator. The third term produces an effective change in initial conditions. This can be seen as follows. If the initial condition voltage $V_0(0)$ is applied to the integrating capacitor in Fig. 5 for a very short time so that the capacitors C_k representing the dielectric absorption remain uncharged, then

$$i_k(0) = -\frac{V_0(0)}{R_k}, \tag{37}$$

and the integrator output due to the last two terms in (30) becomes

$$V_0(s) = \frac{V_0(0)}{s}\left(1 - \sum_k \frac{C_k/C_\infty}{R_k C_k s + 1}\right) \tag{38}$$

where C^* has been set equal to C_∞ in the last term since doing so will be ignoring second-order error terms. Taking the inverse transformation of (38) yields

$$V_0(t) = V_0(0)\left[1 - \frac{V_r(t)}{V_B}\right] \tag{39}$$

where $V_r(t)$ is the recovery voltage following an applied voltage V_B. If the capacitors C_k are initially charged from a previous operation or by applying the initial conditions for a longer time, the recovery voltage will be different from that defined by (21). This effective change in initial conditions would be of particular significance in high-speed repetitive computers where high accuracy is required and where the initial conditions must be reset in minimum time. It would also be important for boundary value problems in which the solution depends critically on the initial, or boundary, conditions.

Errors in Solutions of Equations

Let the computer be set up to solve a differential equation described by the characteristic equation

$$C\{s\} = 0. \tag{40}$$

If only integrators and summers are used and if each integrator output is described by (30), it is clear that the operator s in the given characteristic equation will be replaced by $sC^*(s)/C_\infty$. That is, the characteristic equation solved by the computer will be

$$C\left\{s\frac{C^*(s)}{C_\infty}\right\} = 0. \tag{41}$$

Then, if s_i is a root of (40) and s_i' is a root of (41),

$$s_i = s_i' \frac{C^*(s_i')}{C_\infty}. \tag{42}$$

Let the roots of the computer solution be given by

$$s_i' = s_i + e_i \tag{43}$$

where e_i is the error in the root s_i. Substituting (43) in (42) yields

$$e_i \cong -s_i \sum_k \frac{C_k/C_\infty}{1 + T_k s_i} \tag{44}$$

where it is assumed that $e_i \ll s_i$. From (44) the errors in the characteristic roots can be evaluated if the roots of the given equation and the capacitor dielectric properties are known.

Example of Simple Harmonic Motion

As an example of the computer error produced by dielectric absorption, consider the equation of simple harmonic motion:

$$\ddot{x} + \omega^2 x = 0. \tag{45}$$

The roots of the characteristic equation are

$$s_i = \pm j\omega. \tag{46}$$

From (44), (43), and (19), the roots of the computer solution are

$$s_i' = -\omega\delta \pm j\omega\left(1 - \sum_k \frac{C_k/C_\infty}{1 + \omega^2 T_k^2}\right). \tag{47}$$

Thus the dielectric absorption introduces damping in the solution and changes the frequency of oscillation. The damping ratio resulting from dielectric absorption is a function of frequency and is approximately

$$\zeta = \delta(\omega). \tag{48}$$

It was shown in an earlier article[11] that, if the computer is set up as shown in Fig. 6, the damping ratio introduced by computer bandwidth limitations is

$$\zeta = \frac{1}{\omega T_1} - \frac{\omega(T_s + 2T_2)}{2} \tag{49}$$

where T_1 and T_2 are time constants determined by the integrator characteristics and T_s is the time constant determined by the summer characteristics. The total damping ratio is the sum of (48) and (49).

Experimental Results

To compare the theoretical damping ratio given by (48) and (49) with that obtained experimentally, the Sterling Electronic Differential Analyzer[12] was used. The computer was set up as shown in Fig. 6. The capacitors used in the integrators were the type for which experimental values were given in Table I. The integrator and summer time constants were determined by using the methods given in the previous article[11] and were

$$\frac{1}{T_1} = 5.2 \times 10^{-6}$$

$$T_2 = 2.0 \times 10^{-7}$$

$$T_s = 2.0(1 + \omega^2) \times 10^{-7}. \qquad (50)$$

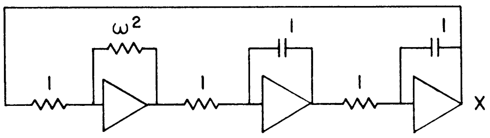

Fig. 6—Computer setup for simple harmonic motion.

The theoretical damping ratio vs frequency is shown in Fig. 7, and the portion due to dielectric absorption only is indicated. The experimentally observed damping ratio is plotted in Fig. 7 and shows very good agreement with the predicted values.

[12] Model LM-10, Sterling Instruments Co., Detroit, Mich.

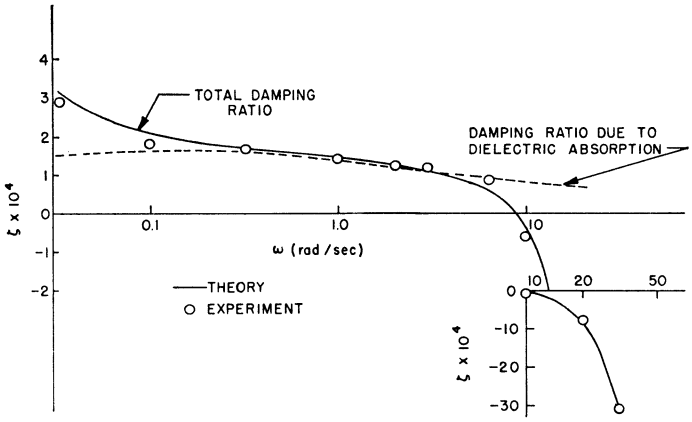

Fig. 7—Damping ratio introduced by the computer.

It will be noted that, for frequencies between 0.1 and 10 rad/second, the dielectric absorption makes the major contribution to the damping. It is clearly a source of error which must not be overlooked when considering the accuracy of differential analyzers, especially at low frequencies.

Acknowledgment

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