Errors in Series/Parallel Resistor Reconnection (Hamon Resistor)

Textbooks and journals often omit the detailed calculations required to adequately demonstrate some widely used concept due to lack of space. In PDF form there is no such limit. Here we consider 3 resistors, wired first in series, then in parallel, and the resistance ratio this represents. The Fluke 752A Reference Divider is used throughout as an example of a practical implementation of this method.

Some readers may feel that the number of calculation steps is excessive. Such people can jump several steps at a time in order to not be bored. Others may appreciate the extra steps. It is difficult, as a reader, to request the intermediate calculation steps; they have been included to cater for all readers.

Initially we will only consider the simple case of the resistors being connected with zero resistance joints or links, and no shunt leakage resistance anywhere.

We further initially neglect important imperfections in the resistors such as: temperature coefficient of resistance (TCR), voltage coefficient of resistance (VCR), and power coefficient of resistance (PCR). All of these are *very* significant when reaching for the highest levels of accuracy.

This technical note was inspired by a thread on the EEVblog Metrology forum.[1]

CONTENTS

[1] "Influence of switch resistance in Hamon Dividers". There were significant contributions to the understanding and analysis on this thread from the users *e61_phil*, *Dr. Frank*, and *Kleinstein*.

3 Resistors in Series

We consider that these 3 resistors all deviate from their average value by a small per-unit amount.

$$R_1 = R_A(1+\delta_1)$$
$$R_2 = R_A(1+\delta_2)$$
$$R_3 = R_A(1+\delta_3)$$

But $R_A \equiv \dfrac{R_1 + R_2 + R_3}{3} = \dfrac{1}{3}\left[R_A(1+\delta_1) + R_A(1+\delta_2) + R_A(1+\delta_3)\right]$

$$R_A = R_A + \frac{R_A}{3}\left[\delta_1 + \delta_2 + \delta_3\right]$$

$$\frac{R_A}{3}\left[\delta_1 + \delta_2 + \delta_3\right] = 0$$

$\therefore \delta_1 + \delta_2 + \delta_3 = 0$ ***The sum of deltas is zero***, which we will use later.

The series resistance is: $R_S = R_1 + R_2 + R_3 = 3R_A$

The following (non-standard) notation will be useful as we proceed:

Write $\Delta \equiv \delta_1 + \delta_2 + \delta_3$ ***the sum of deltas***

Write $\Pi \equiv \delta_1\delta_2 + \delta_2\delta_3 + \delta_3\delta_1$ ***the product of deltas***

The *three-bar equals sign* can be read as "is defined as" or "is identically equal to".

3 Resistors in Parallel

The parallel resistance is
$$R_P = \left[\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right]^{-1}$$

$$R_P = \left[\frac{1}{R_A(1+\delta_1)} + \frac{1}{R_A(1+\delta_2)} + \frac{1}{R_A(1+\delta_3)}\right]^{-1}$$

$$R_P = R_A\left[\frac{1}{(1+\delta_1)} + \frac{1}{(1+\delta_2)} + \frac{1}{(1+\delta_3)}\right]^{-1}$$

$$R_P = R_A\left[\frac{(1+\delta_2)(1+\delta_3)+(1+\delta_1)(1+\delta_3)+(1+\delta_1)(1+\delta_2)}{(1+\delta_1)(1+\delta_2)(1+\delta_3)}\right]^{-1}$$

$$R_P = R_A\left[\frac{(1+\delta_1)(1+\delta_2)(1+\delta_3)}{(1+\delta_2)(1+\delta_3)+(1+\delta_1)(1+\delta_3)+(1+\delta_1)(1+\delta_2)}\right]$$

This is as far as the analysis goes in ***Calibration: Philosophy in Practice*** (2nd Ed, 1994) by Fluke. It should be noted that their equation actually uses resistances for the delta terms, and then (unconsciously) redefines them as per-unit values in their 'significance' table!

$$R_P = \frac{R_A}{3} \times (1+\varepsilon)$$

It is the value of ε that we wish to evaluate as a function of the deltas.

$$\frac{1+\varepsilon}{3} = \left[\frac{(1+\delta_1)(1+\delta_2)(1+\delta_3)}{(1+\delta_2)(1+\delta_3)+(1+\delta_1)(1+\delta_3)+(1+\delta_1)(1+\delta_2)}\right]$$

Multiply out the top of the square bracket:

$$[top] = (1+\delta_1)(1+\delta_2)(1+\delta_3) = (1+\delta_1+\delta_2+\delta_1\delta_2)(1+\delta_3)$$
$$[top] = (1+\delta_1+\delta_2+\delta_1\delta_2)+(\delta_3+\delta_1\delta_3+\delta_2\delta_3+\delta_1\delta_2\delta_3)$$
$$[top] = 1+(\delta_1+\delta_2+\delta_3)+(\delta_1\delta_2+\delta_1\delta_3+\delta_2\delta_3)+\delta_1\delta_2\delta_3$$
$$[top] = 1+\Delta+\Pi+\delta_1\delta_2\delta_3$$

We know the ***sum of deltas, Δ, is zero***, so:

$$[top] = 1+\Pi+\delta_1\delta_2\delta_3$$

$$[bottom] = (1+\delta_2)(1+\delta_3)+(1+\delta_1)(1+\delta_3)+(1+\delta_1)(1+\delta_2)$$
$$[bottom] = (1+\delta_2+\delta_3+\delta_2\delta_3)+(1+\delta_1+\delta_3+\delta_1\delta_3)+(1+\delta_1+\delta_2+\delta_1\delta_2)$$

$$[bottom] = 3+2\Delta+\Pi$$

We know the ***sum of deltas, Δ, is zero***, so:

$$[bottom] = 3+\Pi$$

$$\frac{[top]}{[bottom]} = \frac{1+\Pi+\delta_1\delta_2\delta_3}{3+\Pi}$$

$$\frac{[top]}{[bottom]} = \frac{1}{3}\times\left[\frac{1+\Pi+\delta_1\delta_2\delta_3}{1+\Pi/3}\right]$$

$$1+\varepsilon = \left[\frac{1+\Pi+\delta_1\delta_2\delta_3}{1+\Pi/3}\right]$$

$$1+\varepsilon=\frac{\left(1+\Pi/3\right)+\left(2\Pi/3+\delta_1\delta_2\delta_3\right)}{1+\Pi/3}$$

The third-order term in deltas can be neglected as it is necessarily much smaller than the second-order terms.

$$\varepsilon \cong \frac{2}{3}\Pi$$

But the three delta terms are not independent since the ***sum of deltas is zero***:

$$\delta_1+\delta_2+\delta_3=0$$

We can therefore eliminate one of them, $\delta_3=-\left(\delta_1+\delta_2\right)$

$$\varepsilon \cong \frac{2}{3}\left(\delta_1\delta_2+\delta_1\delta_3+\delta_2\delta_3\right)=\frac{2}{3}\left[\delta_1\delta_2+\delta_3\left(\delta_1+\delta_2\right)\right]=\frac{2}{3}\left[\delta_1\delta_2-\left(\delta_1+\delta_2\right)^2\right]$$

$$\varepsilon \cong -\frac{2}{3}\left(\delta_1^2+\delta_1\delta_2+\delta_2^2\right)$$

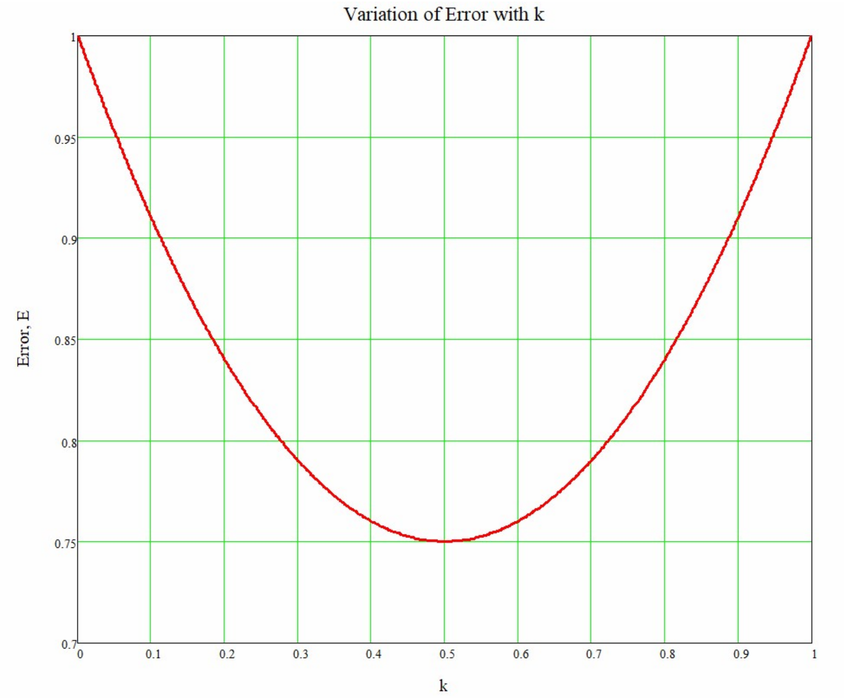
We wish to make a calculation based on the biggest deviation. We can arbitrarily (but without loss of generality) label the resistor with the biggest deviation as R_1. This makes $\left|\delta_1\right|$ the largest value. Having made this decision, δ_1 and δ_2 cannot have the same sign, as that would make $\left|\delta_3\right|$ the largest value.

δ_1 and δ_2 therefore have opposite signs.

We can then write $\delta_2=-k\cdot\delta_1$ for $0\le k\le 1$

Put $E=\left(\delta_1^2+\delta_1\delta_2+\delta_2^2\right)=\left(\delta_1^2-k\cdot\delta_1^2+k^2\delta_1^2\right)=\delta_1^2\left(1-k+k^2\right)$

(*Differentiating doesn't help in this case, as all it does is find the minimum*.)



Putting ***k***=0 is the same as putting ***k***=1 as far as the error is concerned. We either make $\delta_2 = 0$ or $\delta_3 = 0$. In any case the third-order delta term becomes zero in the maximum error case, reinforcing our decision to neglect it.

$$\varepsilon = -\frac{2\delta_1^2}{3}$$

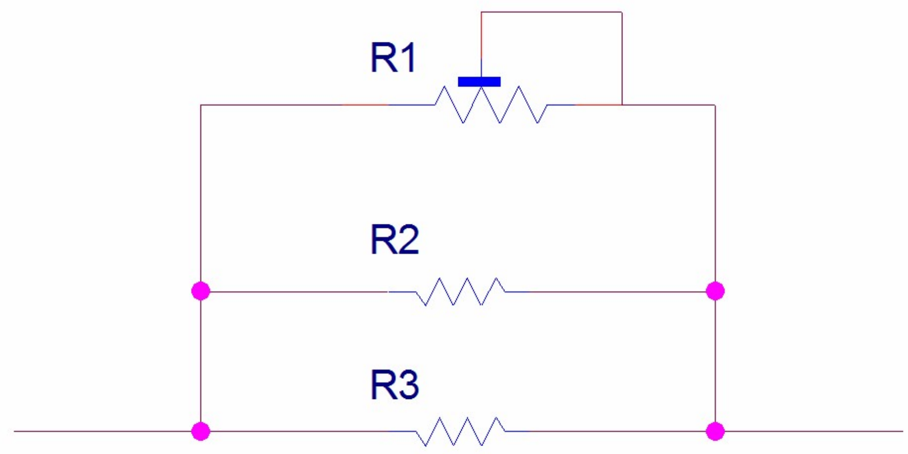
$$R_P = \frac{R_A}{3} \times \left(1 - \frac{2}{3}\delta_1^2\right) \qquad \frac{R_S}{R_P} = 9 \times \left(1 - \frac{2}{3}\delta_1^2\right)$$

largest deviation from average value	resultant deviation from the ideal ratio
1000 ppm	0.667 ppm
300 ppm	0.060 ppm
100 ppm	0.007 ppm
40 ppm	0.001 ppm

The values in this table become very 'theoretical' quite quickly because other errors rapidly become dominant.

Adjustment of the Parallel Resistance

As part of the Fluke 752A's operation, the parallel resistance is adjusted to be equal to a reference value by adjustment of just one of the three paralleled resistors. The manual claims that having done so, the series resistance is necessarily an *exact* ratio to the reference value.



$$R_1 = R_A(1+\delta_1) \qquad R_2 = R_A(1+\delta_2) \qquad R_3 = R_A(1+\delta_3)$$

Since we are using the same notation as for the earlier case with 3 resistors in parallel, we can jump straight into the previous calculation, but making sure **not** to set $\Delta = 0$.

$$R_P = R_A \times \left(\frac{1+\varepsilon}{3}\right)$$

$$\frac{1+\varepsilon}{3} = \frac{1+\Delta+\Pi+\delta_1\delta_2\delta_3}{3+2\Delta+\Pi}$$

Neglecting the third-order term

$$1+\varepsilon = \frac{(1+2\Delta/3+\Pi/3)+(\Delta/3+2\Pi/3)}{1+2\Delta/3+\Pi/3}$$

$$1+\varepsilon = 1+\left(\frac{\Delta/3+2\Pi/3}{1+2\Delta/3+\Pi/3}\right)$$

$$\varepsilon \cong \frac{\Delta+2\Pi}{3}$$

The simple trick of making the sum of deltas zero does not make this error term *exactly* zero. In fact, to make this error term exactly zero we have to make the sum of deltas not quite equal to zero.

The second-order delta terms do not occur in the series configuration at all, so adjusting the parallel error term to zero is inconvenient, mathematically speaking. On the other hand, we know that the second-order delta terms contribute very little to the parallel resistance, provided the resistors are well matched to start with.

But actually we have a slightly different problem. It is not the matching of the resistors within the set of three that is the issue; it is their matching to the reference resistance that is relevant. In the Fluke 752A, the reference resistance is notionally 3 off 120 K resistors in parallel. It means that we could consider selecting 6 near identical resistors, 3 for the reference resistor, and 3 for the series/parallel divider resistors.[2]

$$R_1 = R_A(1+\delta_1) \qquad R_2 = R_A(1+\delta_2) \qquad R_3 = R_A(1+\delta_3)$$

$$R_S = R_1 + R_2 + R_3 = 3R_A\left(1+\frac{\delta_1+\delta_2+\delta_3}{3}\right)$$

$$R_S = 3R_A\left[1+\frac{\Delta}{3}\right] \qquad\qquad R_P = \frac{R_A}{3}\left[1+\frac{\Delta}{3}+\frac{2}{3}\Pi\right]$$

$$\frac{R_S}{R_P} = \frac{3R_A\left[1+\frac{\Delta}{3}\right]}{\frac{R_A}{3}\left[1+\frac{\Delta}{3}+\frac{2}{3}\Pi\right]} = 9\times\left(1-\frac{2\Pi}{3}\right)$$

Our earlier simplification of saying that *the sum of deltas equals zero* is seen to be unnecessary, as that term cancels out anyway.

[2] The reality is of course more complicated than this. The 752A contains all sorts of selectable linked resistors to do the coarse trimming of the resistances.

We now have a new problem to solve. For a given $\frac{\Delta}{3}$, required to match the parallel network to the 40 K reference resistance, what is the worst case value of $\frac{2\Pi}{3}$?

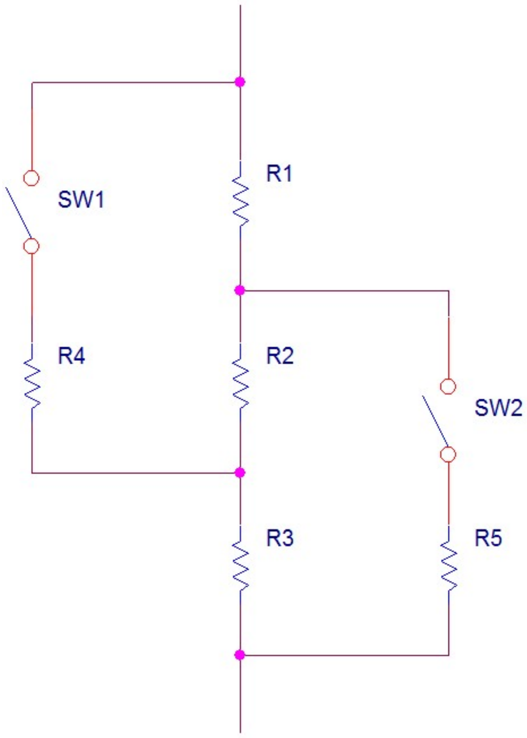
It turns out that for the worst case, all the deltas have the same magnitude as $\frac{\Delta}{3}$, two being of opposite sign to the other.

average deviation from reference value ($\Delta/3$)	sum of deltas (Δ)	resultant deviation from the ideal ratio ($2\Pi/3$)
1000 ppm	3000 ppm	6.000 ppm
333 ppm	1000 ppm	0.667 ppm
100 ppm	300 ppm	0.060 ppm
57 ppm	171 ppm	0.020 ppm
33 ppm	100 ppm	0.007 ppm

We conclude that the resistors need to be selected very carefully so that the adjustment required is less than 100 ppm. This way the second-order delta term error drops below 0.06 ppm, and whilst not negligible, it does become acceptable.

The Fluke 752A User Manual (clause 3-31) claims that resistor mismatch error "is negligible due to the close matching performed in the factory". Of course this may no longer be true after 10 years of service.

Switching from Series to Parallel

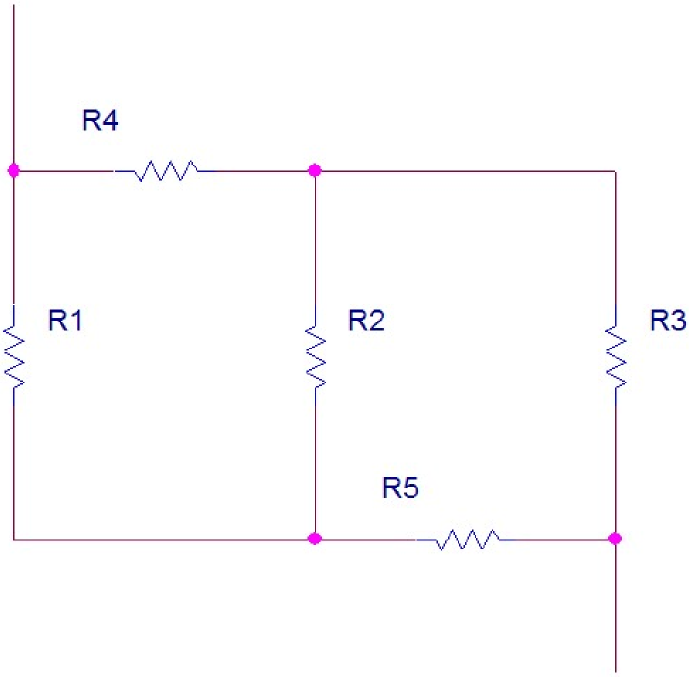


Sticking with the Fluke 752A scheme, it is not immediately obvious that the two switch contacts connect R1, R2, and R3 in parallel. Re-drawing the circuit helps with this understanding.

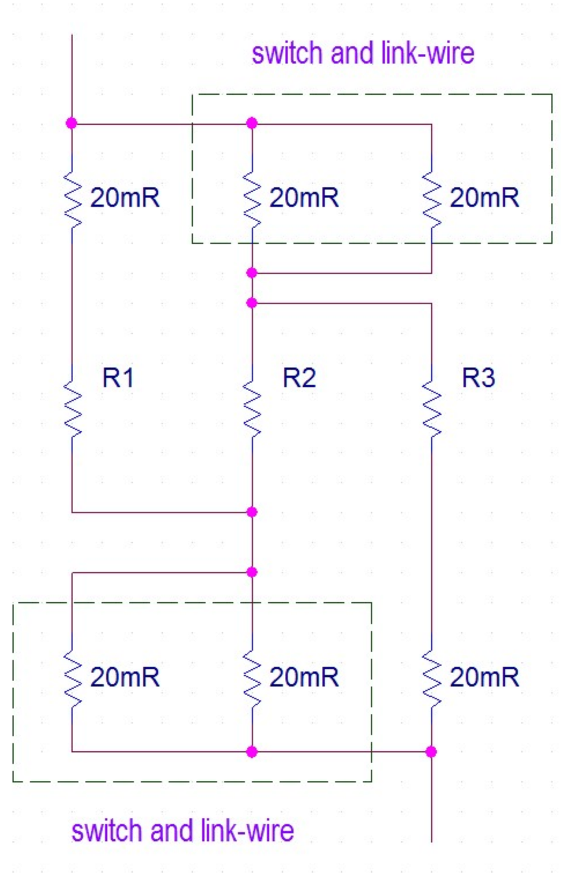
The switches SW1 and SW2 are considered as perfect; the switch contact resistances are simply lumped into R4 and R5.

The (perfect) switches have not been shown in this simplified drawing of the parallel connection. The switch resistance is perhaps around 10 mΩ, but this is not an insignificant amount.

Notice that the series wiring resistance is effectively absorbed into the values of R1, R2, and R3.



It is possible to accurately model this circuit with a circuit simulator, provided that the accuracy settings within the simulator are adjusted appropriately. R4 and R5 include not only the switch resistances, but also the link-wires necessary to join the nets together. To get a more intuitive model of what is happening we can re-draw the circuit again.



Here, the 10 mΩ switch and link-wire resistance has been split into two equal parallel paths, and the switch connection to R1 has been moved back along the interconnect wiring by 20 mΩ. Effectively each of the three paralleled resistors has gained 20 mΩ at each end. The calculation has been greatly simplified in the process.

$$R_S = R_1 + R_2 + R_3 + 2\times 20m\Omega$$
$$= 3R_A + 4\times R_{SW}$$

$$R_P \cong \frac{R_A + 2\times 20m\Omega}{3}$$
$$= \frac{R_A + 4\times R_{SW}}{3}$$

Write $$R_{SW} = \delta_{SW}\cdot R_A$$

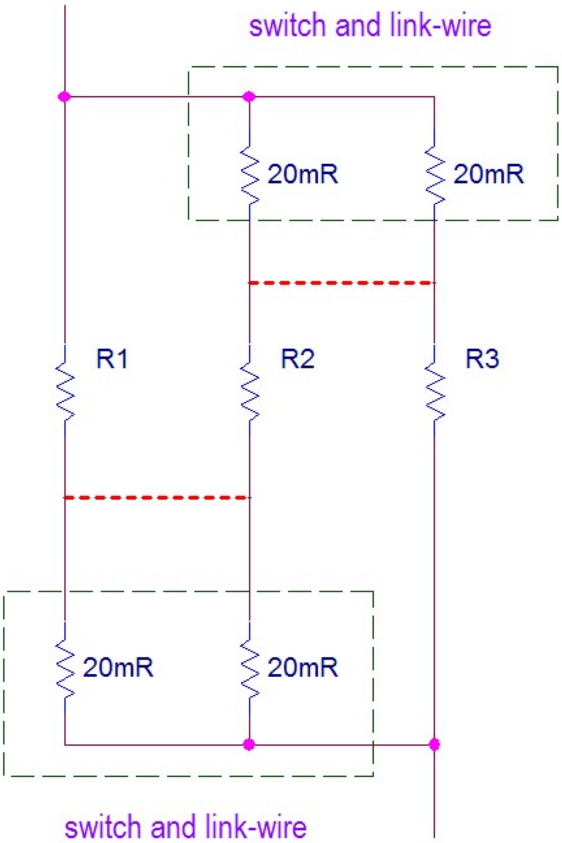
$$\frac{R_S}{R_P} = \frac{3R_A\left(1+\frac{4}{3}\delta_{SW}\right)}{\frac{R_A}{3}\left(1+4\delta_{SW}\right)} = 9\times\left(\frac{1+4\delta_{SW}-4\delta_{SW}+\frac{4}{3}\delta_{SW}}{1+4\delta_{SW}}\right) \cong 9\times\left(1-2.7\delta_{SW}\right)$$

The figures from the Fluke 752A manual state 10 mΩ switch resistance, and 120 kΩ nominal for resistors R1, R2, and R3.

Hence $\delta_{SW} = \dfrac{0.01}{120\times 10^3} = 0.083\,ppm$ (nominal).

This makes the ratio error 0.22 ppm, which is bigger than the accuracy quoted for the overall equipment.[3] However, there is also nothing to say that Fluke have actually added in the extra resistance which we used to simplify the calculation.

[3] *e61_phil* calculated a ratio error of 0.22 ppm for the correct circuit (without the extra resistances shown above) by direct computation, and confirmed by LTSpice, as the very first post in the EEVblog thread.



The 10 mΩ switch and link-wire resistances have been split into two equal parallel paths again, but this time we realise that there is (almost) no current flowing in the dotted red wires, so we can cut them (when looking at the parallel configuration)!

The series resistance is unchanged:

$$R_S = R_1 + R_2 + R_3 = 3R_A$$

R1 and R3 each have an extra 20 mΩ in series with them, whilst R2 has an extra 40 mΩ.

The calculation of the parallel resistance is now a bit more complicated.

$$R_P = \left[\frac{1}{R_A + 20m\Omega} + \frac{1}{R_A + 40m\Omega} + \frac{1}{R_A + 20m\Omega} \right]^{-1}$$

$$R_P = \left[\frac{1}{R_A + 2 \times R_{SW}} + \frac{1}{R_A + 4 \times R_{SW}} + \frac{1}{R_A + 2 \times R_{SW}} \right]^{-1}$$

$$R_P = \left[\frac{1}{R_A (1 + 2\delta_{SW})} + \frac{1}{R_A (1 + 4\delta_{SW})} + \frac{1}{R_A (1 + 2\delta_{SW})} \right]^{-1}$$

$$R_P = R_A \left[(1 - 2\delta_{SW}) + (1 - 4\delta_{SW}) + (1 - 2\delta_{SW}) \right]^{-1}$$

$$R_P = \frac{R_A}{3 - 8\delta_{SW}} = \frac{R_A}{3} \times \frac{1}{1 - \frac{8}{3}\delta_{SW}}$$

$$\frac{R_S}{R_P} = \frac{3R_A}{\left(\frac{R_A}{3} \times \frac{1}{1-2.7\delta_{SW}}\right)} = 9 \times \left(1 - 2.7\delta_{SW}\right)$$

Which is the same ratio we got when we added in the extra interconnect wiring to simplify the network problem.[4]

The Fluke manual claims that the worst case effect of the switch resistance is actually only 0.042 ppm after adjusting the interconnection resistances.[5]

In effect they are saying that the switch resistance is stable to a factor of

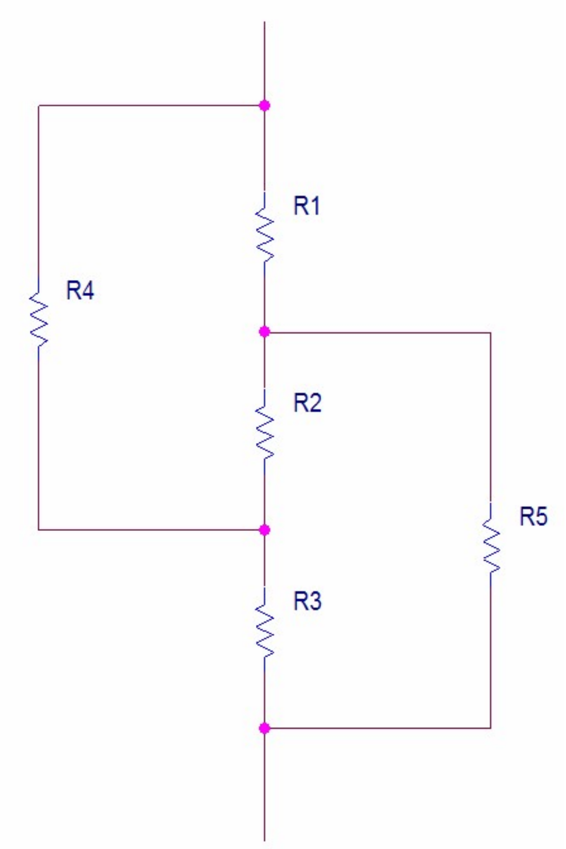
0.042 ppm / 0.22 ppm = 0.19

or ±20% variation. This amounts to ±2 mΩ, although at no point do they suggest measuring or checking it as part of the routine maintenance. This is another unseen and unverified error source, which potentially makes the reference divider inaccurate without the user knowing about it.

[4] *Kleinstein*, post #43, gets the same result by considering that only 2/3 of the main current flows through each of the switch resistances. The resulting volt drops are then averaged.

[5] *Dr Frank*, post #48, locates the compensation as a wire, noted on the circuit diagram as “36.75 inches long AWG 22 solid copper”.

Leakage Resistance



If we just consider the resistor chain in isolation, the parallel configuration has no internal leakage nodes. The series configuration has two leakage nodes, and we show them here with leakage resistances, R4 and R5, across the open series/parallel switch contacts.

The errors due to leakage are necessarily small, so we can consider the effect of R4 and R5 separately. We can also consider the resistors R1, R2, and R3 as nominal values, as again their deviation from nominal is not significant for this leakage analysis.

$$R_1 = R_2 = R_3 = R_A \qquad R_4 \equiv k \cdot R_A \qquad R_5 \equiv k \cdot R_A$$

Effect of R5: $R_S = 3R_A$ becomes

$$R_S = R_A + \left[\frac{1}{2R_A} + \frac{1}{kR_A}\right]^{-1} = R_A\left(1 + \left[\frac{k+2}{2k}\right]^{-1}\right) = R_A\left(1 + \left[\frac{2k}{k+2}\right]\right)$$

$$R_S = R_A\left(1 + \frac{(2k+4)-4}{k+2}\right) = R_A\left(3 - \frac{4}{k+2}\right) \cong 3R_A\left(1 - \frac{4}{3k}\right)$$

R4 and R5 are in similar positions, so they have the same effect.

Effect of both R4 and R5:

$$R_S = 3R_A\left(1-\frac{8}{3k}\right) = 3R_A(1-\varepsilon)$$

required error due to leakage	required *k* multiplier	resultant required leakage resistance
1.000 ppm	2.7×10^6	> 0.32 TΩ
0.300 ppm	8.9×10^6	> 1.1 TΩ
0.100 ppm	2.7×10^7	> 3.2 TΩ
0.030 ppm	8.9×10^7	> 11 TΩ
0.010 ppm	2.7×10^8	> 32 TΩ

We have shown the effect of leakage resistances across the switch contacts. Depending on the exact implementation of the design, it is also possible for the leakage currents to flow to ground. In this case a new analysis would be required.

Certainly there is no shortage of surfaces on which contamination can build up (undetected) over the years. Fortunately, the 100:1 attenuator is using a driven guard to reduce the leakage currents.

This leakage is another error source which is essentially impossible to evaluate on a complete system which has been in service for some time.

Voltage Coefficient / Power Coefficient

The whole basis of the design is that by wiring the resistors in series, and then in parallel, we get a mathematically well defined ratio between these two configurations. But this can only strictly be true for ideal resistors, with no switch resistance, and no leakage resistance.

In the calibration mode the 120 K resistors are driven with 10 V, a power dissipation of 0.83 mW. When used at 100 V, each resistor then sees 30 V, a power dissipation of 7.5 mW.

Consider a 100 K UPW25B100KV precision wire-wound resistor for this application. Using its 0.02% derating curve, we have 60% of 250 mW (= 150 mW) at 80°C, falling linearly to zero at 115°C. We estimate 150 mW/35°C, which suggests a self-heating of 1°C/4.3 mW.

The 7.5 mW gives a 1.7°C rise, and at 5 ppm/°C this gives an 8.5 ppm shift.

In this application there is no ability to use a tracking TC to boost the performance. The resistors have to have absolute performance. The Fluke manual claims a maximum of 0.05 ppm due to this error source.

1°C rise at 0.05 ppm/°C

0.1°C rise at 0.5 ppm/°C

0.05°C rise at 1 ppm/°C

If we go to the Precision Resistor Company we can get ±1 ppm/°C TC specification (on special request). Looking at the 0.1% derating curve, we have 50% power falling over 10°C. For a 2.5 W HR5032N resistor, this is 1250 mW/10°C, or 1°C/125 mW self-heating. It means the 7.5 mW gives 7.5/125 = 0.06°C rise. It means we can just about achieve the stated specification due to self-heating.

Their data sheet does not mention voltage coefficient, which in any case is indistinguishable from power coefficient for low frequency use.

The good news is that if the attenuation was verified at the time of manufacture then there is no reason to suppose that the TC or self-heating effect will change over the entire service life. Nevertheless, we have to *believe* the original calibration, since it is not possible to verify this performance without a higher accuracy calibration standard.

The power coefficient effect can be reduced, for a given resistor type, by simply using series/parallel resistor combinations. For example a 120 K resistor made from 4 off 120 K resistors in series/parallel should be expected to have a power coefficient of resistance at least 4× better than the basic performance. It is “at least” because if the temperature coefficients match inexactly, there may be some statistical improvement. However, there is a technical downside, in addition to the increased cost and space.

A (2-wire) precision resistor has a low TC resistance element and copper lead wires. Suppose the resistor has a ±5 ppm/°C temperature coefficient. The lead wires, being copper, have a TC of 0.4%/°C = 4000 ppm/°C. The lead-wire resistance needs to be less than 0.1% of the main resistance value in order to not adversely contribute to the overall TC.

If you make a resistor out of say 100 similar resistors (as 10 paralleled chains of 10 series elements), the fractional lead-wire resistance is unchanged. Making longer chains, with fewer chains in parallel, however, increases the lead-resistance fraction. Making a network of 10 parallel chains of 10 series resistors, there are also now 90 leakage nodes. This is clearly more demanding for the low leakage mounting points.

For Vishay AUR foil resistors, the best TC is down to 0.05 ppm/°C. The 300 mW power derating curve falls to zero at 150°C from 70°C. This gives 1°C/3.75 mW, and a 2°C rise for this application. It is 2x worse than the requirement. But worse than that, the voltage coefficient is stated as < 0.1 ppm/V.[6] The requirement here is 0.05 ppm/30 V or 0.0017 ppm/V. The Vishay Foil marketing brochure claims that the voltage coefficient is "non-measurable".

Vishay also quote a convenient Power Coefficient of Resistance (PCR), for example on their VSA101 resistors. ±5 ppm at rated power (600 mW) suggests 0.06 ppm in this application.

Closing Remarks

The Fluke 752A Reference Divider is well engineered to produce a tight linearity specification. However, it encompasses several design 'features' which make it unsuitable as a primary standard, and it cannot be adequately verified by its own self-calibration procedures.

These uncalibratable features include:

- The voltage coefficient and power coefficient of the 120 K series/parallel resistors need to be accepted on faith.
- The switch resistance variation is not considered.
- If the 40 K reference resistance drifts significantly relative to the 120 K resistors, balancing the primary chain can create a significant mismatch, changing the series/parallel ratio, and hence the 10:1 ratio accuracy.
- The leakage resistance is kept adequately low by sealing the box and by cleaning, but is not verifiable by measurement.

The Fluke 752A achieves its impressive performance, not by virtue of a clever technical principal, but simply by excellent attention to detail in its design.

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[6] The Vishay RCK series data sheet notes that this is the resolution limit of the test equipment, and "essentially zero".