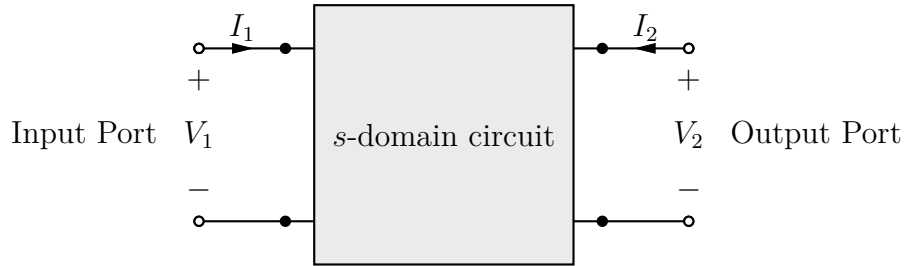


Veritasium's video

ABCD parameters of a lossless transmission line



Let L and C , respectively, denote the inductance and capacitance per unit length of a lossless transmission line. The s -domain ABCD two-port description of a lossless transmission, is given by

$$\begin{aligned} V_1(s) &= V_2(s) \cosh(\gamma l) + I_2(s) Z_o \sinh(\gamma l) \\ I_1(s) &= \frac{V_2(s)}{Z_o} \sinh(\gamma l) + I_2(s) \cosh(\gamma l), \end{aligned}$$

where

$$\begin{aligned} Z_o &= \sqrt{\frac{L}{C}}, \\ \gamma &= s\sqrt{LC} = s\frac{1}{c_s}, \end{aligned} \tag{1}$$

where l is the length of the line and $c_s = \frac{1}{\sqrt{LC}}$ is the propagation velocity in the transmission line.

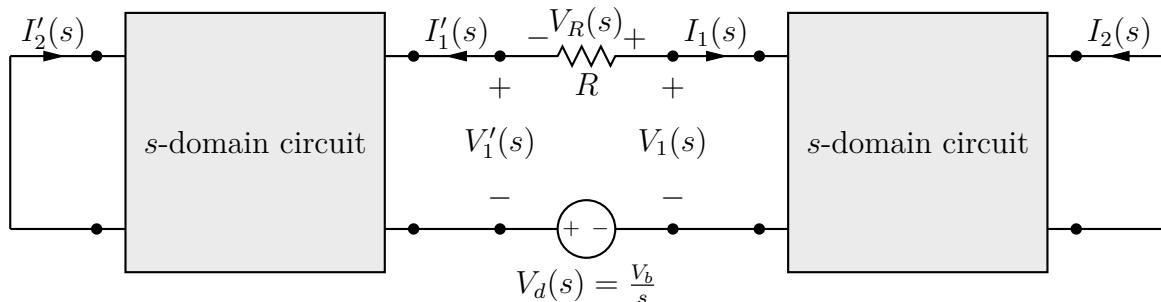
In Veritasium's video the two transmission lines are both terminated in short circuits, i.e. $V_2(s) = 0$. The equations above now reduce to:

$$V_1(s) = I_2(s) Z_o \sinh(\gamma l) \tag{2}$$

$$I_1(s) = I_2(s) \cosh(\gamma l). \tag{3}$$

The problem

The figure below shows Veritasium's problem with the two transmission lines replaced by their two-port equivalent circuits. Note that the circuit on the left has been flipped around its vertical centre line.



By applying Equations (2) and (3) to the two two-port circuits, we get

$$V_1(s) = I_2(s)Z_o \sinh(\gamma l) \quad (4)$$

$$I_1(s) = I_2(s) \cosh(\gamma l), \quad (5)$$

$$V_1'(s) = I_2'(s)Z_o \sinh(\gamma l) \quad (6)$$

$$I_1'(s) = I_2'(s) \cosh(\gamma l). \quad (7)$$

From the circuit diagram, we see that

$$I_1'(s) = -I_1(s). \quad (8)$$

According to Equations (5) and (7):

$$I_2(s) \cosh(\gamma l) = -I_2'(s) \cosh(\gamma l),$$

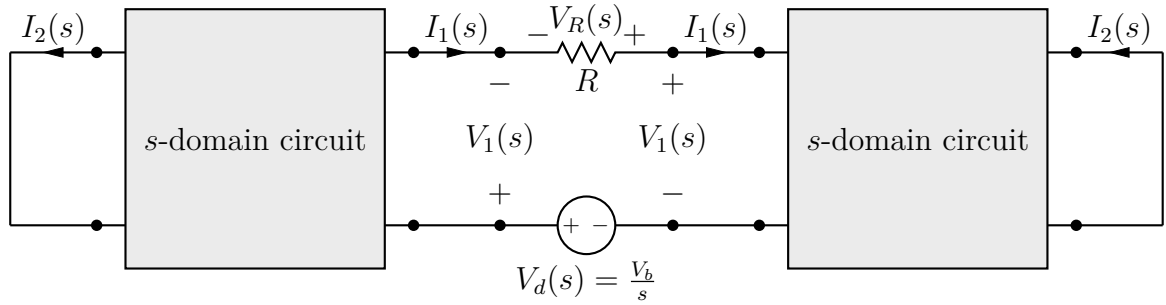
which tells us that

$$I_2'(s) = -I_2(s). \quad (9)$$

Furthermore, according to Equations (4) and (6):

$$V_1'(s) = -V_1(s). \quad (10)$$

Having this information, we can now redraw the circuit:



By applying Kirchoff's voltage law to the inner loop, we get

$$V_1(s) + RI_1(s) + V_1(s) - V_d(s) = 0, \quad (11)$$

which simplifies to

$$V_d(s) = RI_1(s) + 2V_1(s). \quad (12)$$

By using equations (4) and (5)

$$\begin{aligned} V_d(s) &= RI_1(s) + 2I_2(s)Z_o \sinh(\gamma l) \\ &= RI_1(s) + 2Z_o I_1(s) \frac{\sinh(\gamma l)}{\cosh(\gamma l)}. \end{aligned}$$

According to the definitions of sinh and cosh:

$$V_d(s) = RI_1(s) + 2Z_o I_1(s) \frac{1 - e^{-2\gamma l}}{1 + e^{-2\gamma l}} \quad (13)$$

$$V_d(s)(1 + e^{-2\gamma l}) = RI_1(s)(1 + e^{-2\gamma l}) + 2Z_o I_1(s)(1 - e^{-2\gamma l}). \quad (14)$$

Define $\tau = 2l\sqrt{LC}$. It follows from Equation (1) that τ is the time that it takes for the electromagnetic wave to propagate from the source to the end of the line, and back to the source. In this example τ is 1 second.

Equation (14) is now rewritten in terms of τ :

$$V_d(s)(1 + e^{-2\tau s}) = RI_1(s)(1 + e^{-\tau s}) + 2Z_o I_1(s)(1 - e^{-\tau s}).$$

Let's take the inverse Laplace transform:

$$v_d(t) + v_d(t - \tau) = Ri_1(t) + Ri_1(t - \tau) + 2Z_o i_1(t) - 2Z_o i_1(t - \tau). \quad (15)$$

Since the switch is open for $t < 0$ s, both $v_d(t) = 0$ V and $i_1(t) = 0$ A for $t < 0$ s. This also tells us that both $v_d(t - \tau) = 0$ V and $i_1(t - \tau) = 0$ A for $t < \tau$.

So for $0 \leq t < \tau$, Equation (15) simplifies to:

$$v_d(t) = Ri_1(t) + 2Z_o i_1(t). \quad (16)$$

By solving for $i_1(t)$, we get

$$i_1(t) = \frac{v_d(t)}{R + 2Z_o}. \quad (17)$$

and

$$v_R(t) = v_d(t)i_1(t) = v_d(t) \frac{R}{R + 2Z_o} \quad (18)$$

This equation tells us that for $0 \leq t < \tau$, the battery voltage is divided by the series combination of the lamp resistance R and characteristic impedance of the two transmission lines in series, i.e. $2Z_o$. This is true, irrespective of the waveshape of $v_d(t)$. This tells us that the transmission behaves like a resistor with value of Z_o before the first reflection comes back at time τ .

For $t > \tau$, Equation (15) describes the interaction between the incident and reflected waves at the start of the two transmission lines.

Step response

In Veritasium's video, a battery in series with a switch forms the voltage source $v_d(t)$. This source is modelled as

$$v_d(t) = V_b u(t),$$

where V_b is the DC battery voltage and $u(t)$ is the unit step function. In the s -domain this translates to

$$V_d(s) = \frac{V_b}{s}.$$

By iteratively applying Equation (15), we see that $I_1(t)$ will be a step-like waveform with each step occurring at time $t = n\tau$, for $n = 0, 1, 2, \dots$

We can now use discrete-time methods to analyse this behaviour in more detail. We sample the analogue quantities every τ seconds, **just after** the step occurred.

Taking the z -transform of Equation (15), we get

$$V_d(z) + z^{-1}V_d(z) = R(I_1(z) + z^{-1}I_1(z)) + 2Z_o(I_1(z) - z^{-1}I_1(z)).$$

Solving for $I_1(z)$ gives

$$I_1(z) = V_d(z) \frac{1 + z^{-1}}{R + 2Z_o + (R - 2Z_o)z^{-1}}. \quad (19)$$

Since $v_d(t) = V_b u(t)$, the z -transform of $v_d(t)$ is

$$V_d(z) = \frac{V_b}{1 - z^{-1}}.$$

Equation (19) now becomes

$$I_1(z) = V_b \frac{1 + z^{-1}}{(R + 2Z_o + (R - 2Z_o)z^{-1})(1 - z^{-1})}.$$

Decomposing into partial fractions, yields:

$$\begin{aligned} I_1(z) &= V_b \frac{\frac{1}{R}}{(1 - z^{-1})} - V_b \frac{\frac{2Z_o}{R}}{R + 2Z_o - z^{-1}(2Z_o - R)} \\ &= V_b \frac{\frac{1}{R}}{(1 - z^{-1})} - V_b \frac{\frac{2Z_o}{R2(Z_o+R)}}{1 - \frac{2Z_o-R}{2Z_o+R}z^{-1}} \end{aligned}$$

By taking the inverse z -transform we get:

$$i(n\tau) = \frac{V_b}{R} - \frac{2Z_o}{R(2Z_o + R)} \left[\frac{2Z_o - R}{2Z_o + R} \right]^n V_b, \quad (20)$$

for $n = 0, 1, 2, \dots$

(Note we made the choice that $0^0 = 1$.)

Comparison with Lapi's simulation

Lapi used the following parameters in his simulation:

Parameter	Value
C	3.66 pF m ⁻¹
L	3.04 μH m ⁻¹
V_b	1 V
R	50 Ω

Given these parameters, the characteristic impedance Z_o of the line is equal to

$$Z_o = \sqrt{\frac{L}{C}} = 911.37\Omega$$

and a propagation speed c_s is

$$c_s = \frac{1}{\sqrt{LC}} = 2.9979 \times 10^8 \text{ m s}^{-1}.$$

Plotting the $i(n\tau)$ by making use of Equation (20), gives the result in Figure 1. This **analytic** result agrees very well with Lapi's simulation.

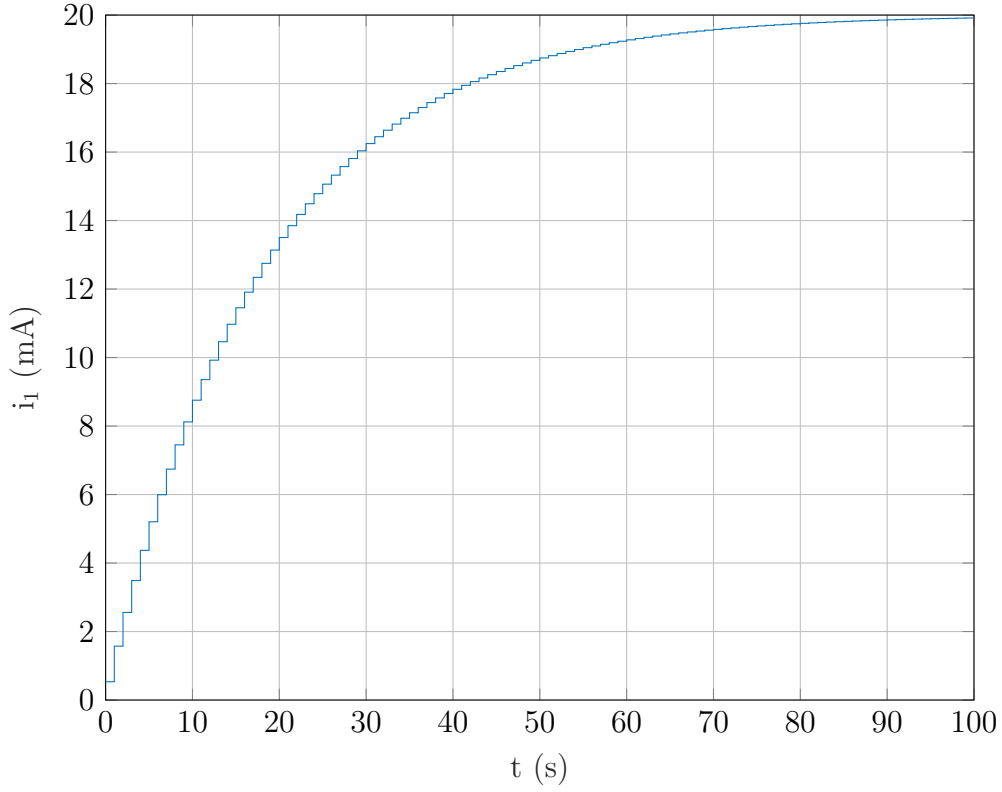


Figure 1: Comparison with Lapi's simulation

There's more

We can learn a lot more by studying Equation (20).

The voltage over the lamp, just after time zero depends heavily on the ratio of R to $2Z_o$.

1. If $R = 2Z_o$, we have perfect series termination. Before $t = 1s$, the battery voltage divides evenly between R and $2Z_o$ and the voltage across the lamp is thus equal to $V_b/2$. After $t = \tau$, the voltage across the lamp is equal to V_b .
2. If $0 < R < 2Z_o$, the term

$$\left[\frac{2Z_o - R}{2Z_o + R} \right]$$

in Equation (20) is positive and less than 1. Since

$$\lim_{n \rightarrow \infty} x^n = 0 \quad \text{for } 0 < x < 1,$$

the steady-state value of $i_1(t)$ is

$$i(n\tau) = \frac{V_b}{R}. \quad (21)$$

3. If $R = 0$, there is no damping in the systems. Current i_1 will increase to infinity over time.
4. If $R > 2Z_o$, the term

$$\left[\frac{2Z_o - R}{2Z_o + R} \right]$$

is negative, but greater than -1 . The current oscillates before it reaches the steady state. An example is shown in Figure 3. Note that the voltage across the lamp is sometimes greater than the battery voltage.

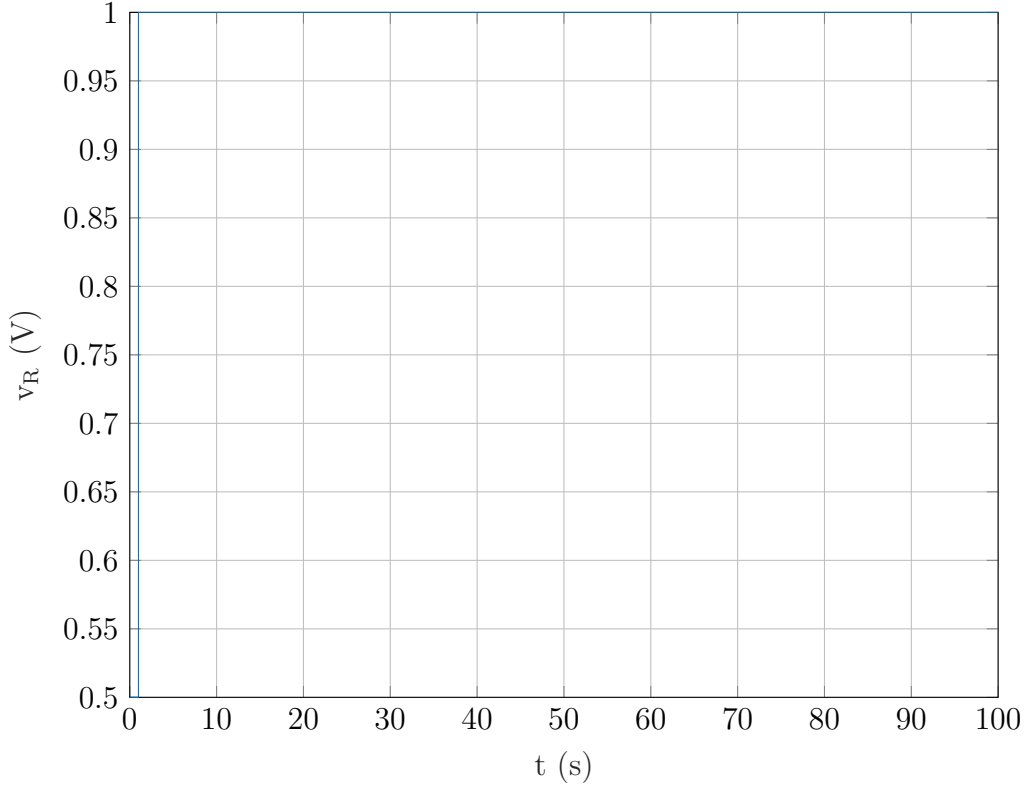


Figure 2: Plot of the voltage across the lamp when $R = 2Z_o$.

Conclusion

This analytic model provides a lot of insight into the behaviour of the voltage and current at the input to the two transmission lines.

However, it does not describe the region surrounding to the battery accurately. The electric and magnetic fields in this region can only be analysed through simulation.

Energy from the battery can only reach the lamp after $\frac{1m}{c}$ s, at the very least. My bet is on "None of the above".

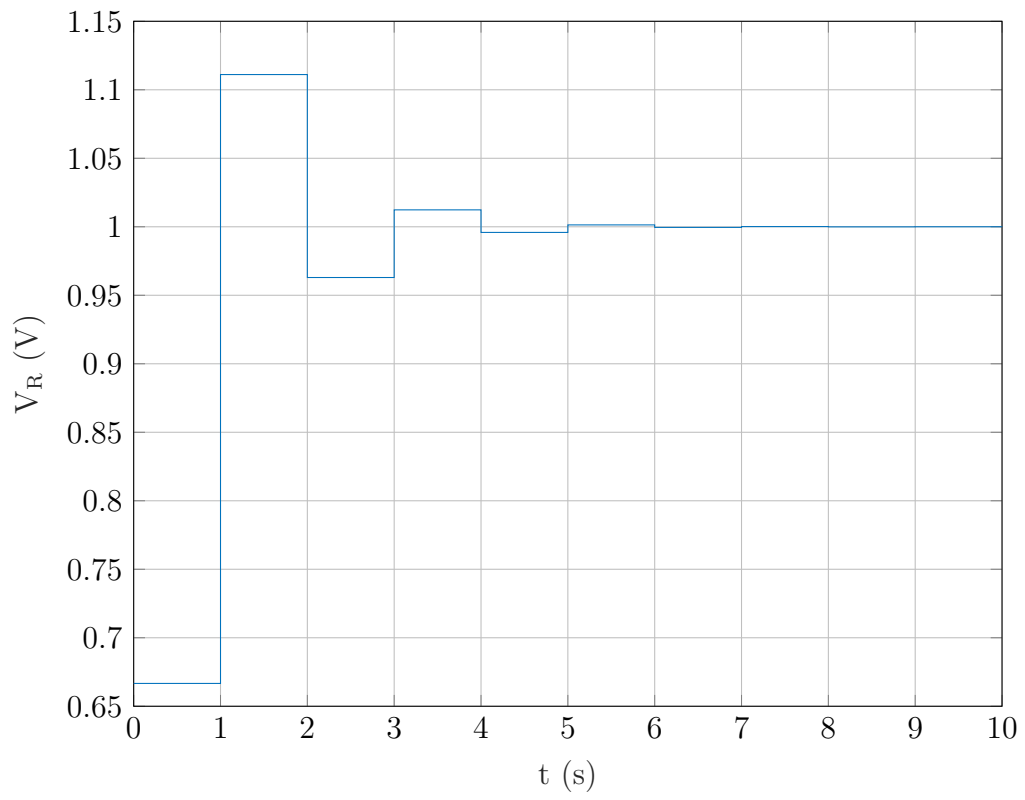


Figure 3: Plot of the voltage across the lamp with $R = 4Z_o$.