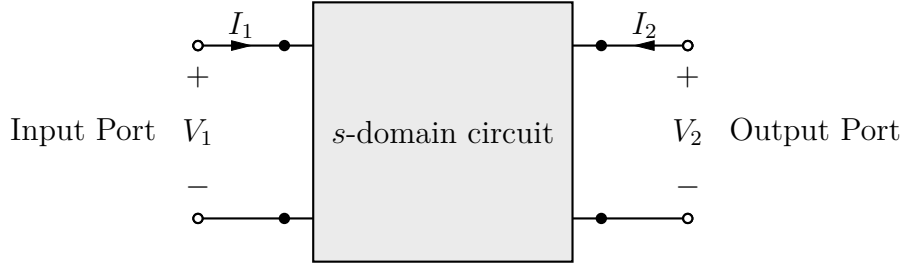


Transmission Line Perspective

ABCD parameters of a lossless transmission line



Let L and C denote the inductance and capacitance per unit length of a lossless transmission line, respectively. The s -domain ABCD two-port description of a lossless transmission, is given by

$$\begin{aligned} V_1(s) &= V_2(s) \cosh(\gamma l) + I_2(s) Z_o \sinh(\gamma l) \\ I_1(s) &= \frac{V_2(s)}{Z_o} \sinh(\gamma l) + I_2(s) \cosh(\gamma l), \end{aligned}$$

where

$$\begin{aligned} Z_o &= \sqrt{\frac{L}{C}}, \\ \gamma &= s\sqrt{LC} = s\frac{1}{c_s}, \end{aligned} \tag{1}$$

where l is the length of the line and $c_s = \frac{1}{\sqrt{LC}}$ is the propagation velocity in the transmission line. Resistance Z_o is the characteristic impedance of the transmission line and γ is the propagation constant.

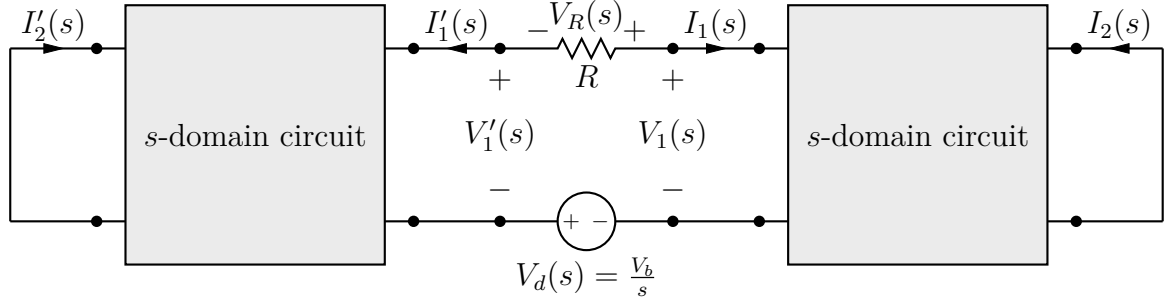
The two transmission lines are both terminated in short circuits, i.e. $V_2(s) = 0$. The equations above now reduce to:

$$V_1(s) = I_2(s) Z_o \sinh(\gamma l) \tag{2}$$

$$I_1(s) = I_2(s) \cosh(\gamma l). \tag{3}$$

The problem

The figure below shows the problem with the two transmission lines replaced by their two-port equivalent circuits. Note that the circuit on the left has been flipped around its vertical centre line.



By applying Equations (2) and (3) to the two two-port circuits, we get

$$V_1(s) = I_2(s)Z_o \sinh(\gamma l) \quad (4)$$

$$I_1(s) = I_2(s) \cosh(\gamma l), \quad (5)$$

$$V'_1(s) = I'_2(s)Z_o \sinh(\gamma l) \quad (6)$$

$$I'_1(s) = I'_2(s) \cosh(\gamma l). \quad (7)$$

From the circuit diagram, we see that

$$I'_1(s) = -I_1(s). \quad (8)$$

According to Equations (5) and (7):

$$I_2(s) \cosh(\gamma l) = -I'_2(s) \cosh(\gamma l),$$

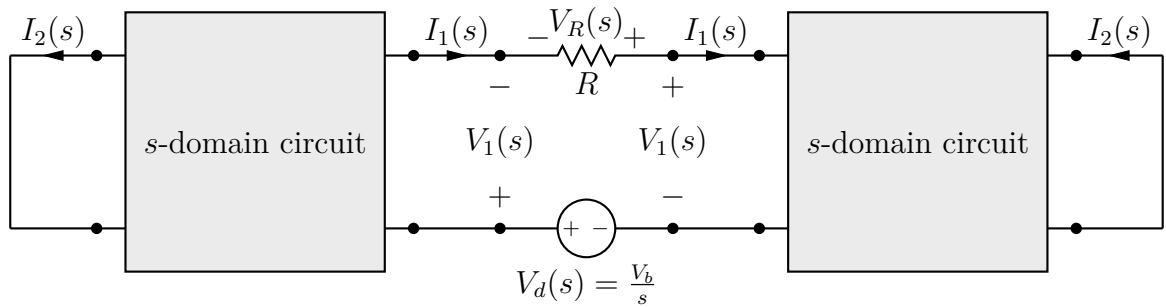
which tells us that

$$I'_2(s) = -I_2(s). \quad (9)$$

Furthermore, according to Equations (4) and (6):

$$V'_1(s) = -V_1(s). \quad (10)$$

Having this information, we can now redraw the circuit:



By applying Kirchoff's voltage law to the inner loop, we get

$$V_1(s) + RI_1(s) + V_1(s) - V_d(s) = 0, \quad (11)$$

which simplifies to

$$V_d(s) = RI_1(s) + 2V_1(s). \quad (12)$$

By using equations (4) and (5)

$$\begin{aligned} V_d(s) &= RI_1(s) + 2I_2(s)Z_o \sinh(\gamma l) \\ &= RI_1(s) + 2Z_o I_1(s) \frac{\sinh(\gamma l)}{\cosh(\gamma l)}. \end{aligned}$$

According to the definitions of \sinh and \cosh :

$$V_d(s) = RI_1(s) + 2Z_o I_1(s) \frac{1 - e^{-2\gamma l}}{1 + e^{-2\gamma l}} \quad (13)$$

$$V_d(s)(1 + e^{-2\gamma l}) = RI_1(s)(1 + e^{-2\gamma l}) + 2Z_o I_1(s)(1 - e^{-2\gamma l}). \quad (14)$$

Define $\alpha = 2l\sqrt{LC}$. It follows from Equation (1) that α is the time that it takes for the electromagnetic wave to propagate from the source to the end of the line, and back to the source.

Equation (14) is now rewritten in terms of α :

$$V_d(s)(1 + e^{-2\alpha s}) = RI_1(s)(1 + e^{-\alpha s}) + 2Z_o I_1(s)(1 - e^{-\alpha s}).$$

Let's take the inverse Laplace transform:

$$v_d(t) + v_d(t - \alpha) = Ri_1(t) + Ri_1(t - \alpha) + 2Z_o i_1(t) - 2Z_o i_1(t - \alpha). \quad (15)$$

Since the switch is open for $t < 0$ s, both $v_d(t) = 0$ V and $i_1(t) = 0$ A for $t < 0$ s. This also tells us that both $v_d(t - \alpha) = 0$ V and $i_1(t - \alpha) = 0$ A for $t < \alpha$.

So for $0 \leq t < \alpha$, Equation (15) simplifies to:

$$v_d(t) = Ri_1(t) + 2Z_o i_1(t). \quad (16)$$

By solving for $i_1(t)$, we get

$$i_1(t) = \frac{v_d(t)}{R + 2Z_o}. \quad (17)$$

and

$$v_R(t) = v_d(t)i_1(t) = v_d(t) \frac{R}{R + 2Z_o} \quad (18)$$

This equation tells us that for $0 \leq t < \alpha$, the battery voltage is divided by the series combination of the lamp resistance R and characteristic impedance of the two transmission lines in series, i.e. $2Z_o$. This is true, irrespective of the shape of $v_d(t)$. This tells us that the transmission behaves like a resistor with value of Z_o before the first reflection comes back at time α .

For $t > \alpha$, Equation (15) describes the interaction between the incident and reflected waves at the start of the two transmission lines.

Step response

In Veritasium's video, a battery in series with a switch forms the voltage source $v_d(t)$. This source is modelled as

$$v_d(t) = V_b u(t),$$

where V_b is the DC battery voltage and $u(t)$ is the unit step function. In the s -domain this translates to

$$V_d(s) = \frac{V_b}{s}.$$

By iteratively applying Equation (15), we see that $I_1(t)$ will be a step-like waveform with each step occurring at time $t = n\alpha$, for $n = 0, 1, 2, \dots$

We can now use discrete-time methods to analyse this behaviour in more detail. We sample the analogue quantities every α seconds, **just after** the step occurred.

Taking the z -transform of Equation (15), we get

$$V_d(z) + z^{-1}V_d(z) = R(I_1(z) + z^{-1}I_1(z)) + 2Z_o(I_1(z) - z^{-1}I_1(z)).$$

Solving for $I_1(z)$ gives

$$I_1(z) = V_d(z) \frac{1 + z^{-1}}{R + 2Z_o + (R - 2Z_o)z^{-1}}. \quad (19)$$

Since $v_d(t) = V_b u(t)$, the z -transform of $v_d(t)$ is

$$V_d(z) = \frac{V_b}{1 - z^{-1}}.$$

Equation (19) now becomes

$$I_1(z) = V_b \frac{1 + z^{-1}}{(R + 2Z_o + (R - 2Z_o)z^{-1})(1 - z^{-1})}.$$

Decomposing into partial fractions, yields:

$$\begin{aligned} I_1(z) &= V_b \frac{\frac{1}{R}}{(1 - z^{-1})} - V_b \frac{\frac{2Z_o}{R}}{R + 2Z_o - z^{-1}(2Z_o - R)} \\ &= V_b \frac{\frac{1}{R}}{(1 - z^{-1})} - V_b \frac{\frac{2Z_o}{R(2Z_o + R)}}{1 - \frac{2Z_o - R}{2Z_o + R}z^{-1}} \end{aligned}$$

By taking the inverse z -transform we get:

$$i(n\alpha) = \frac{V_b}{R} - \frac{2Z_o}{R(2Z_o + R)} \left[\frac{2Z_o - R}{2Z_o + R} \right]^n V_b, \quad (20)$$

for $n = 0, 1, 2, \dots$

(Note we made the choice that $0^0 = 1$.)

According to Eq. (20), $i(t)$ is a stepped exponential function with time constant

$$\tau = \frac{\alpha}{\ln \left[\frac{2Z_o - R}{2Z_o + R} \right]}. \quad (21)$$

Comparison COMSOL's simulation

Fig. 1 shows a comparison between COMSOL's simulation and transmission line theory. The time constant $\tau = 717\text{ns}$.

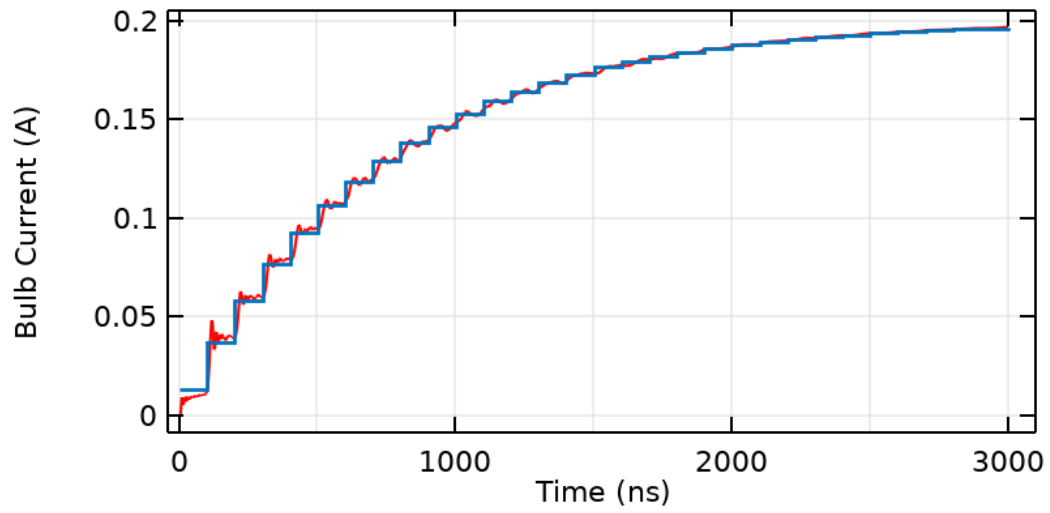


Figure 1: Comparison with COMSOL's simulation. The red curve is COMSOL's simulation and the blue curve is from transmission line theory.

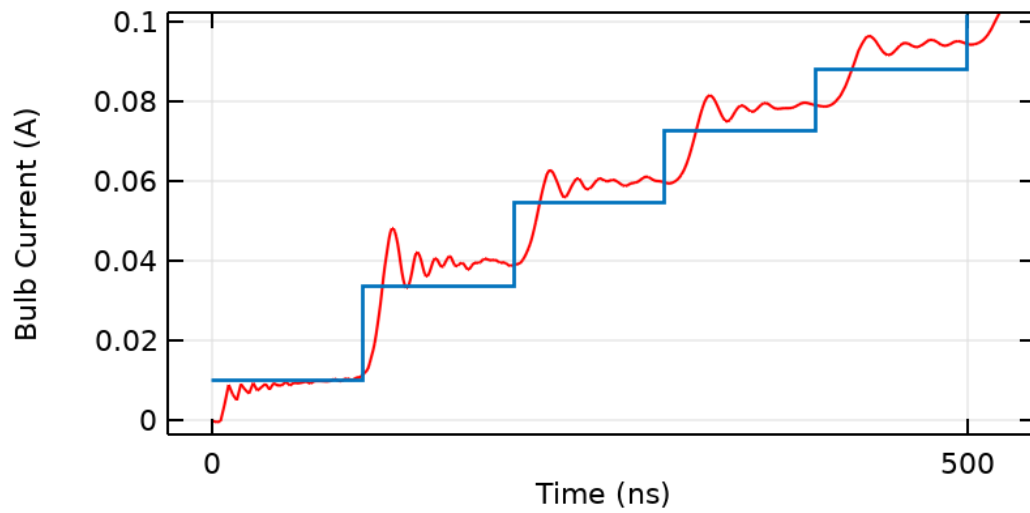


Figure 2: Comparison with COMSOL's simulation. The red curve is COMSOL's simulation and the blue curve is from transmission line theory.

The characteristic impedance of the line is $Z_0 = 359\Omega$. By choosing the resistance R of the bulb equal to $2Z_0$, the voltage across the bulb would be equal to half the battery voltage after 3.3ns.

Fig. 2 is similar to Fig. 1 and shows the results over the first 500ns.