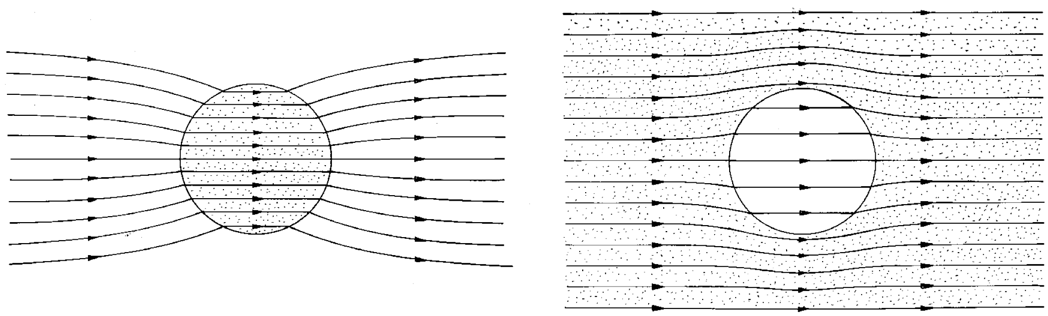

Fig. 7.5.2 Distribution of current density in and around the rod of Fig. 7.5.1. (a) $\sigma_b \geq \sigma_a$. (b) $\sigma_a \geq \sigma_b$.

regions, they have very different behaviors where the conductivity is discontinuous. In fact, the normal component of the current density is continuous at the interface, and the spacing between lines of **J** must be preserved across the interface. Thus, in the distribution of *current density* shown in Fig. 7.5.2, the lines are continuous. Note that the current tends to concentrate on the rod if it is more conducting, but is diverted around the rod if it is more insulating.

A surface charge density resides at the interface between the conducting media of different conductivities. This surface charge density acts as the source of **E** on the cylindrical surface and is identified by (7.2.17).

Inside-Outside Approximations. In exploiting the formal analogy between fields in linear dielectrics and in Ohmic conductors, it is important to keep in mind the very different physical phenomena being described. For example, there is no conduction analog to the free space permittivity ϵ_o. There is no minimum value of the conductivity, and although ϵ can vary between a minimum of ϵ_o in free space and $1000\epsilon_o$ or more in special solids, the electrical conductivity is even more widely varying. The ratio of the conductivity of a copper wire to that of its insulation exceeds 10^{21}.

Because some materials are very good conductors while others are very good insulators, steady conduction problems can exemplify the determination of fields for large ratios of physical parameters. In Sec. 6.6, we examined field distributions in cases where the ratios of permittivities were very large or very small. The "inside-outside" viewpoint is applicable not only to approximating fields in dielectrics but to finding the fields in the transient EQS systems in the latter part of this chapter and in MQS systems with magnetization and conduction.

Before attempting a more general approach, consider the following example, where the fields in and around a resistor are described.

Example 7.5.2. Fields in and around a Conductor

The circular cylindrical conductor of Fig. 7.5.3, having radius b and length L, is surrounded by a perfectly conducting circular cylindrical "can" having inside

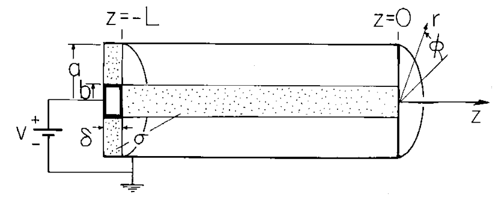

Fig. 7.5.3 Circular cylindrical conductor surrounded by coaxial perfectly conducting "can" that is connected to the right end by a perfectly conducting "short" in the plane $z = 0$. The left end is at potential v relative to right end and surrounding wall and is connected to that wall at $z = -L$ by a washer-shaped resistive material.

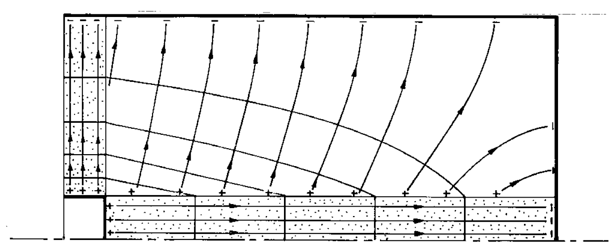

Fig. 7.5.4 Distribution of potential and electric field intensity for the configuration of Fig. 7.5.3.

radius a. With respect to the surrounding perfectly conducting shield, a dc voltage source applies a voltage v to the perfectly conducting disk. A washer-shaped material of thickness δ and also having conductivity σ is connected between the perfectly conducting disk and the outer can. What are the distributions of Φ and $\mathbf{E}$ in the conductors and in the annular free space region?

Note that the fields within each of the conductors are fully specified without regard for the shape of the can. The surfaces of the circular cylindrical conductor are either constrained in potential or bounded by free space. On the latter, the normal component of $\mathbf{J}$, and hence of $\mathbf{E}$, is zero. Thus, in the language of Sec. 7.4, the potential is constrained on S' while the normal derivative of Φ is constrained on the insulating surfaces S''. For the center conductor, S' is at $z = 0$ and $z = -L$ while S'' is at $r = b$. For the washer-shaped conductor, S' is at $r = b$ and $r = a$ and S'' is at $z = -L$ and $z = -(L + \delta)$. The theorem of Sec. 7.4 shows that the potential inside each of the conductors is uniquely specified. Note that this is true regardless of the arrangement outside the conductors.

In the cylindrical conductor, the solution for the potential that satisfies Laplace's equation and all these boundary conditions is simply a linear function of z.

$$\Phi^b = -\frac{v}{L} z \tag{6}$$

Thus, the electric field intensity is uniform and z directed.

$$\mathbf{E}^b = \frac{v}{L} \mathbf{i_z} \tag{7}$$

These equipotentials and $\mathbf{E}$ lines are sketched in Fig. 7.5.4. By way of reinforcing what is new about the insulating surface boundary condition, note that (6) and (7) apply to the cylindrical conductor regardless of its cross-section geometry and its length. However, the longer it is, the more stringent is the requirement that the annular region be insulating compared to the central region.

In the washer-shaped conductor, the axial symmetry requires that the potential not depend on z. If it depends only on the radius, the boundary conditions on the insulating surfaces are automatically satsfied. Two solutions to Laplace's equation are required to meet the potential constraints at $r = a$ and $r = b$. Thus, the solution is assumed to be of the form

$$\Phi^c = A ln r + B \tag{8}$$

The coefficients A and B are determined from the radial boundary conditions, and it follows that the potential within the washer-shaped conductor is

$$\Phi^c = v \frac{ln\left(\frac{r}{a}\right)}{ln\left(\frac{b}{a}\right)} \tag{9}$$

The "inside" fields can now be used to determine those in the insulating annular "outside" region. The potential is determined on all of the surface surrounding this region. In addition to being zero on the surfaces $r = a$ and $z = 0$, the potential is given by (6) at $r = b$ and by (9) at $z = -L$. So, in turn, the potential in this annular region is uniquely determined.

This is one of the few problems in this book where solutions to Laplace's equation that have both an r and a z dependence are considered. Because there is no ϕ dependence, Laplace's equation requires that

$$\left(\frac{\partial^2}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}\right)\Phi = 0 \tag{10}$$

The linear dependence on z of the potential at $r = b$ suggests that solutions to Laplace's equation take the product form $R(r)z$. Substitution into (10) then shows that the r dependence is the same as given by (9). With the coefficients adjusted to make the potential $\Phi_a(a, -L) = 0$ and $\Phi_a(b, -L) = v$, it follows that in the outside insulating region

$$\Phi^a = \frac{v}{ln\left(\frac{a}{b}\right)} ln\left(\frac{r}{a}\right)\frac{z}{L} \tag{11}$$

To sketch this potential and the associated **E** lines in Fig. 7.5.4, observe that the equipotentials join points of the given potential on the central conductor with those of the same potential on the washer-shaped conductor. Of course, the zero potential surface is at $r = a$ and at $z = 0$. The lines of electric field intensity that originate on the surfaces of the conductors are perpendicular to these equipotentials and have tangential components that match those of the inside fields. Thus, at the surfaces of the finite conductors, the electric field in region (a) is neither perpendicular nor tangential to the boundary.

For a positive potential v, it is clear that there must be positive surface charge on the surfaces of the conductors bounding the annular insulating region. Remember that the normal component of **E** on the conductor sides of these surfaces is zero. Thus, there is a surface charge that is proportional to the normal component of **E** on the insulating side of the surfaces.

$$\sigma_s(r = b) = \epsilon_o E_r^a(r = b) = -\frac{\epsilon_o v}{b\, ln(a/b)}\frac{z}{L} \tag{12}$$

The order in which we have determined the fields makes it clear that this surface charge is the one required to accommodate the field configuration outside

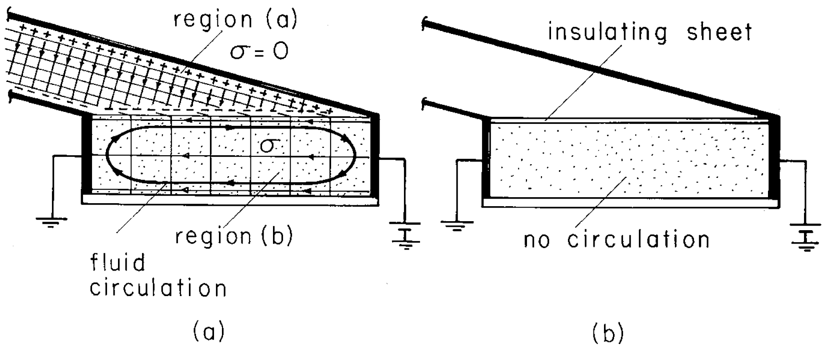

Fig. 7.5.5 Demonstration of the absence of volume charge density and existence of a surface charge density for a uniform conductor. (a) A slightly conducting oil is contained by a box constructed from a pair of electrodes to the left and right and with insulating walls on the other two sides and the bottom. The top surface of the conducting oil is free to move. The resulting surface force density sets up a circulating motion of the liquid, as shown. (b) With an insulating sheet resting on the interface, the circulating motion is absent.

the conducting regions. A change in the shield geometry changes Φ^a but does not alter the current distribution within the conductors. In terms of the circuit analogy used in Sec. 7.0, the potential distributions have been completely determined by the rod-shaped and washer-shaped resistors. The charge distribution is then determined *ex post facto* by the "distributed capacitors" surrounding the resistors.

The following demonstration shows that the unpaired charge density is zero in the volume of a uniformly conducting material and that charges do indeed tend to accumulate at discontinuities of conductivity.

Demonstration 7.5.1. Distribution of Unpaired Charge

A box is constructed so that two of its sides and its bottom are plexiglas, the top is open, and the sides shown to left and right in Fig. 7.5.5 are highly conducting. It is filled with corn oil so that the region between the vertical electrodes in Fig. 7.5.5 is semi-insulating. The region above the free surface is air and insulating compared to the corn oil. Thus, the corn oil plays a role analogous to that of the cylindrical rod in Example 7.5.2. Consistent with its insulating transverse boundaries and the potential constraints to left and right is an "inside" electric field that is uniform.

The electric field in the outside region (a) determines the distribution of charge on the interface. Since we have determined that the inside field is uniform, the potential of the interface varies linearly from v at the right electrode to zero at the left electrode. Thus, the equipotentials are evenly spaced along the interface. The equipotentials in the outside region (a) are planes joining the inside equipotentials and extending to infinity, parallel to the canted electrodes. Note that this field satisfies the boundary conditions on the slanted electrodes and matches the potential on the liquid interface. The electric field intensity is uniform, originating on the upper electrode and terminating either on the interface or on the lower slanted electrode. Because both the spacing and the potential difference vary linearly with horizontal distance, the negative surface charge induced on the interface is uniform.

Wherever there is an unpaired charge density, the corn oil is subject to an electrical force. There is unpaired charge in the immediate vicinity of the interface in the form of a surface charge, but not in the volume of the conductor. Consistent with this prediction is the observation that with the application of about 20 kV to electrodes having 20 cm spacing, the liquid is set into a circulating motion. The liquid moves rapidly to the right at the interface and recirculates in the region below. Note that the force at the interface is indeed to the right because it is proportional to the product of a negative charge and a negative electric field intensity. The fluid moves as though each part of the interface is being pulled to the right. But how can we be sure that the circulation is not due to forces on unpaired charges in the fluid volume?

An alteration to the same experiment answers this question. With a plexiglas sheet placed on the interface, it is mechanically pinned down. That is, the electrical force acting on the unpaired charges in the immediate vicinity of the interface is countered by viscous forces tending to prevent the fluid from moving tangential to the solid boundary. Yet because the sheet is insulating, the field distribution within the conductor is presumably unaltered from what it was before.

With the plexiglas sheet in place, the circulations of the first experiment are no longer observed. This is consistent with a model that represents the corn-oil as a uniform Ohmic conductor[1]. (For a mathematical analysis, see Prob. 7.5.3.)

In general, there is a two-way coupling between the fields in adjacent uniformly conducting regions. If the ratio of conductivities is either very large or very small, it is possible to calculate the fields in an "inside" region ignoring the effect of "outside" regions, and then to find the fields in the "outside" region. The region in which the field is first found, the "inside" region, is usually the one to which the excitation is applied, as illustrated in Example 7.5.2. This will be further illustrated in the following example, which pursues an approximate treatment of Example 7.5.1. The exact solutions found there can then be compared to the approximate ones.

Example 7.5.3. Approximate Current Distribution around Relatively Insulating and Conducting Rods

Consider first the field distribution around and then in a circular rod that has a small conductivity relative to its surroundings. Thus, in Fig. 7.5.1, $\sigma_a \gg \sigma_b$. Electrodes far to the left and right are used to apply a uniform field and current density to region (a). It is therefore in this inside region outside the cylinder that the fields are first approximated.

With the rod relatively insulating, it imposes on region (a) the approximate boundary condition that the normal current density, and hence the radial derivative of the potential, be zero at the rod surface, where $r = R$.

$$\mathbf{n} \cdot \mathbf{J}^a \approx 0 \Rightarrow \frac{\partial \Phi^a}{\partial r} \approx 0 \quad \text{at} \quad r = R \tag{13}$$

Given that the field at infinity must be uniform, the potential distribution in region (a) is now uniquely specified. A solution to Laplace's equation that satisfies this condition at infinity and includes an arbitrary coefficient for hopefully satisfying the

[1] See film *Electric Fields and Moving Media*, produced by the National Committee for Electrical Engineering Films and distributed by Education Development Center, 39 Chapel St., Newton, Mass. 02160.