In recognition of the balance between collision forces and electrical forces, the forces of (4) are replaced by $|q_+|\mathbf{E}$ and $-|q_-|\mathbf{E}$, respectively.

$$P_d = N_+|q_+|\mathbf{E}\cdot\mathbf{v}_+ - N_-|q_-|\mathbf{E}\cdot\mathbf{v}_- \tag{5}$$

If, in turn, the velocities are written as the products of the respective mobilities and the macroscopic electric field, (7.1.3), it follows that

$$P_d = (N_+|q_+|\mu_+ + N_-|q_-|\mu_-)\mathbf{E}\cdot\mathbf{E} = \sigma\mathbf{E}\cdot\mathbf{E} \tag{6}$$

where the definition of the conductivity σ (7.1.7) has been used.

The power dissipation density $P_d = \sigma\mathbf{E}\cdot\mathbf{E}$ (watts/m^3) represents a rate of energy loss from the electromagnetic system to the thermal system.

Example 11.3.1. The Poynting Vector of a Stationary Current Distribution

In Example 7.5.2, we studied the electric fields in and around a circular cylindrical conductor fed by a battery in parallel with a disk-shaped conductor. Here we determine the Poynting vector field and explore its spatial relationship to the dissipation density.

First, within the circular cylindrical conductor [region (b) in Fig. 11.3.1], the electric field was found to be uniform, (7.5.7),

$$\mathbf{E}^b = \frac{v}{L}\mathbf{i_z} \tag{7}$$

while in the surrounding free space region, it was [from (7.5.11)]

$$\mathbf{E}^a = -\frac{v}{L\,ln(a/b)}\Big[\frac{z}{r}\mathbf{i_r} + ln(r/a)\mathbf{i_z}\Big] \tag{8}$$

and in the disk-shaped conductor [from (7.5.9)]

$$\mathbf{E}^c = \frac{v}{ln(a/b)}\frac{1}{r}\mathbf{i_r} \tag{9}$$

By symmetry, the magnetic field intensity is ϕ directed. The ϕ component of $\mathbf{H}$ is most easily evaluated from the integral form of Ampère's law. The current density in the circular conductor follows from (7) as $J_o = \sigma v/L$. Then,

$$2\pi r H_\phi = J_o\pi r^2 \rightarrow H_\phi^b = \frac{J_o r}{2}; \qquad r < b \tag{10}$$

$$2\pi r H_\phi = J_o\pi b^2 \rightarrow H_\phi^a = \frac{J_o b^2}{2r}; \qquad b < r < a \tag{11}$$

The magnetic field distribution in the disk conductor is also deduced from Ampère's law. In this region, it is easiest to evaluate the r component of Ampère's differential law with the current density $\mathbf{J}^c = \sigma\mathbf{E}^c$, with $\mathbf{E}^c$ given by (9). Integration of this partial differential equation on z then gives a linear function of z plus

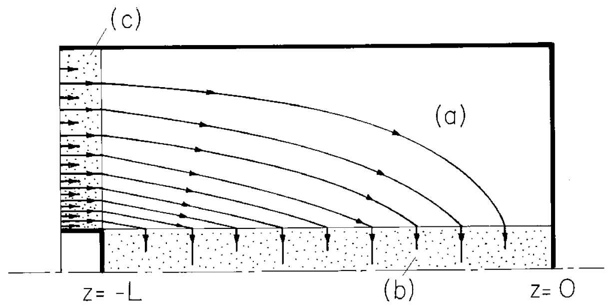

Fig. 11.3.1 Distribution of Poynting flux in coaxial resistors and associated free space. The configuration is the same as for Example 7.5.2. A source to the left supplies current to disk-shaped and circular cylindrical resistive materials. The outer and right-end conductors are perfectly conducting. Note that there is a Poynting flux in the free space interior region even when the currents are stationary.

an "integration constant" that is a function of r. The latter is determined by the requirement that H_ϕ be continuous at $z = -L$.

$$H_\phi^c = -\frac{\sigma}{ln(a/b)}\frac{v}{r}(L+z) + J_o\frac{b^2}{2r}; \qquad b < r < a \tag{12}$$

It follows from these last four equations that the Poynting vector inside the circular cylindrical conductor, in the surrounding space, and in the disk-shaped electrode is

$$\mathbf{S}^b = -\frac{v}{L}\frac{J_o}{2}r\mathbf{i_r} \tag{13}$$

$$\mathbf{S}^a = -\frac{vb^2J_o}{ln(a/b)2rL}\big(\frac{z}{r}\mathbf{i_z} - ln\frac{r}{a}\mathbf{i_r}\big) \tag{14}$$

$$\mathbf{S}^c = \left[\frac{-\sigma}{ln^2(a/b)}\frac{v^2}{r^2}(L+z) + \frac{J_o v}{ln(a/b)}\frac{b^2}{2r^2}\right]\mathbf{i_z} \tag{15}$$

This distribution of **S** is sketched in Fig. 11.3.1. Wherever there is a dissipation density, there must be a negative divergence of **S**. Thus, in the conductors, the **S** lines terminate in the volume. In the free space region (a), **S** is solenoidal. Even with the fields perfectly stationary in time, the power is seen to flow through the *open space* to be absorbed in the volume where the dissipation takes place. The integral of the Poynting vector over the surface surrounding the inner conductor gives what we would expect either from the circuit point of view

$$-\oint \mathbf{E}\times\mathbf{H}\cdot d\mathbf{a} = (2\pi bL)\big(\frac{v}{L}\big)\big(\frac{J_o b}{2}\big) = v(\pi b^2 J_o) = vi \tag{16}$$

where i is the total current through the cylinder, or from an evaluation of the right-hand side of the integral conservation law.

$$\int_V \sigma\mathbf{E}\cdot\mathbf{E}dv = (\pi b^2 L)\sigma\big(\frac{v}{L}\big)^2 = v\big(\pi b^2\sigma\frac{v}{L}\big) = vi \tag{17}$$

An Alternative Conservation Theorem for Electroquasistatic Systems. In describing electroquasistatic systems, it is inconvenient to require that the magnetic field intensity be evaluated. We consider now an alternative conservation theorem that is specialized to EQS systems. We will find an alternative expression for $\mathbf{S}$ that does not involve $\mathbf{H}$. In the process of finding an alternative distribution of $\mathbf{S}$, we *illustrate the danger of ascribing meaning to* $\mathbf{S}$ *evaluated at a point, rather than integrated over a closed surface.*

In the EQS approximation, $\mathbf{E}$ is irrotational. Thus,

$$\mathbf{E} = -\nabla\Phi \tag{18}$$

and the power input term on the left in the integral conservation law, (11.1.1), can be expressed as

$$-\oint_S \mathbf{E}\times\mathbf{H}\cdot d\mathbf{a} = \oint_S \nabla\Phi\times\mathbf{H}\cdot d\mathbf{a} \tag{19}$$

Next, the vector identity

$$\nabla\times(\Phi\mathbf{H}) = \nabla\Phi\times\mathbf{H} + \Phi\nabla\times\mathbf{H} \tag{20}$$

is used to write the right-hand side of (19) as

$$-\oint_S \mathbf{E}\times\mathbf{H}\cdot d\mathbf{a} = \oint_S \nabla\times(\Phi\mathbf{H})\cdot d\mathbf{a} - \oint_S \Phi\nabla\times\mathbf{H}\cdot d\mathbf{a} \tag{21}$$

The first integral on the right is zero because the curl of a vector is divergence free and a field with no divergence has zero flux through a closed surface. Ampère's law can be used to eliminate *curl* $\mathbf{H}$ from the second.

$$-\oint_S \mathbf{E}\times\mathbf{H}\cdot d\mathbf{a} = -\oint_S \Phi\big(\mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}\big)\cdot d\mathbf{a} \tag{22}$$

In this way, we have determined an alternative expression for $\mathbf{S}$, *valid only in the electroquasistatic approximation.*

$$\boxed{\mathbf{S} = \Phi\big(\mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}\big)} \tag{23}$$

The density of power flow, expressed by (23) as the product of a potential and *total current density* consisting of the sum of the conduction and displacement current densities, has a form similar to that used in circuit theory.

The power flux density of (23) is convenient in describing EQS systems, where the effects of magnetic induction are not significant. To be consistent with the EQS approximation, the conservation law must be used with the magnetic energy density neglected.

Example 11.3.2. Alternative EQS Power Flux Density for Stationary Current Distribution

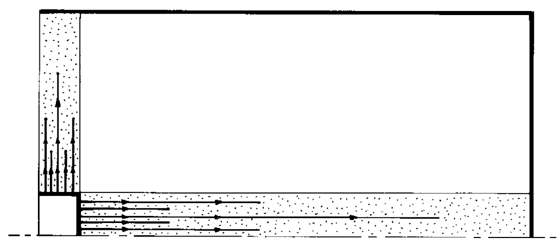

Fig. 11.3.2 Distribution of electroquasistatic flux density for the same system as shown in Fig. 11.3.1.

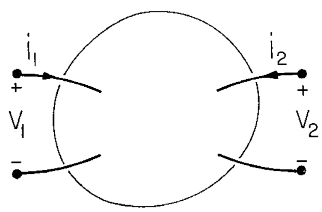

Fig. 11.3.3 Arbitrary EQS system accessed through terminal pairs.

To contrast the alternative EQS power flow density with the Poynting flux density, consider again the coaxial resistor configuration of Example 11.3.1. Because the fields are stationary, the EQS power flux density is

$$\mathbf{S} = \Phi\mathbf{J} \tag{24}$$

By contrast with the Poynting flux density, this vector field is zero in the free space region. In the circular cylindrical conductor, the potential and current density are [(7.5.6) and (7.5.7)]

$$\Phi^b = -\frac{v}{L}z; \qquad \mathbf{J}^b = \sigma\mathbf{E}^b = \frac{\sigma v}{L}\mathbf{i_z} \tag{25}$$

and it follows that the power flux density is simply

$$\mathbf{S} = -\frac{\sigma v^2}{L^2}z\mathbf{i_z} \tag{26}$$

There is a similar, radially directed flux density in the disk-shaped resistor.

The alternative distribution of **S**, shown in Fig. 11.3.2, is clearly very different from that shown in Fig. 11.3.1 for the Poynting flux density.

Poynting Power Density Related to Circuit Power Input. Suppose that the surface S described by the conservation theorem encloses a system that is accessed through terminal pairs, as shown in Fig. 11.3.3. Under what circumstances is the integral of $\mathbf{S} \cdot d\mathbf{a}$ over S equivalent to summing the voltage-current product of the terminals of the wires connected to the system?